

Contributions to the theory of semimodular lattices

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1. Introduction. It is well-known that semimodularity was firstly introduced only for lattices of finite length (G. BIRKHOFF, [1], p. 445). Birkhoff's definition, based on covering property of elements, was not suitable of course to be extended directly for lattices of infinite length¹⁾. Some years later R. CROISOT gave a definition of semimodularity in general, coinciding with the Birkhoff's one in the case of finite length. His definition was formulated in [2] as follows:

Definition 1. A lattice L is called *semimodular* if and only if to each triplet $a, b, x \in L$ with the properties²⁾

$$a \parallel b \quad \text{and} \quad a \wedge b < x < a$$

there exists an element y such that

$$(1) \quad a \wedge b < y \leq b \quad \text{and} \quad (x \vee y) \wedge a = x.$$

One sees at once that the elements a and b do not take symmetric parts in this definition, although they do in Birkhoff's definition. But, as we show in Section 2 of this note, Croisot's definition can be rewritten into a symmetric form. After having proved the equivalence of the new definition to the original one we add some remarks to this definition.

The subject of Section 3 is related to § 4 of [4]. Theorem 6 of that paper shows that there exist complemented semimodular lattices in which no inner element has a maximal or a minimal complement³⁾. Now we deal with lattices satisfying the lower covering condition and we prove a theorem on relative complements from which we can derive, as a special case, the following counterpole of the result quoted just now: In a complemented semimodular lattice each element of finite height has both maximal and minimal complements. In addition, we show that every element of a partition lattice has the same property.

2. On the definition of semimodularity. We rephrase the definition of semimodularity as follows:

¹⁾ For notations and terminology used without being explained in this note, we refer to [3].

²⁾ $a \parallel b$ means that the elements a and b are incomparable.

³⁾ We call the attention of the reader to the fact that the lattice discussed in that theorem is not relatively complemented in general.

Definition 2. A lattice L is called semimodular if and only if to each triplet $a, b, x \in L$ with the properties

$$a \parallel b \quad \text{and} \quad a \wedge b < x \leq a$$

there exists an element z such that

$$(2) \quad a \wedge b < z \leq b,$$

$$(3) \quad (x \vee z) \wedge a = x,$$

$$(4) \quad (x \vee z) \wedge b = z.$$

In order to legitimate this definition we have to prove

Theorem 1. *Definitions 1 and 2 are equivalent over the class of lattices.*

PROOF. Let C_i ($i=1, 2$) denote the class of lattices that are semimodular in the sense of Definition i . Then $C_2 \subseteq C_1$ obviously. Thus we have to verify the reversed inclusion.

Let L be any lattice from the class C_1 and let a, b be any pair of incomparable elements of L . Further, let x denote any element of L such that $a \wedge b < x \leq a$.

In case of $x=a$ we make the choice $z=b$ which meets the requirements (2)–(4) trivially.

In case of $x < a$ there exists an $y \in L$ satisfying (1) by Definition 1. Choose

$$(5) \quad z = (x \vee y) \wedge b.$$

Then

$$b \geq z \geq y \wedge b = y > a \wedge b,$$

verifying the inequalities in (2), and

$$x \vee y \geq z \geq y,$$

implying

$$x \vee y \geq x \vee z \geq x \vee y,$$

whence

$$x \vee y = x \vee z.$$

From this equation we get (3) and (4) by direct calculation:

$$(x \vee z) \wedge a = (x \vee y) \wedge a = x$$

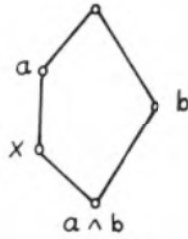
by (1), and

$$(x \vee z) \wedge b = (x \vee y) \wedge b = z$$

by (5). Hence, each lattice L contained in the class C_1 belongs to C_2 . Thus, Theorem 1 is proved.

Remark 1. In the French version of [3] we have pointed out that in case of a modular lattice L the requirements in Definition 1 can be satisfied by choosing $y=b$. Actually, a lattice is modular if and only if (3) can be solved by $z=b$ for each x

satisfying the inequalities $a \frown b < x \leq a$; in fact, any non-modular lattice has a sublattice S represented by the diagram



in which $(x \vee b) \wedge a = a \neq x$.

Remark 2. Moreover, if L is modular, then (3) and (4) are valid with *any* x and z satisfying the inequalities

$$a \frown b \leq x \leq a \quad \text{and} \quad a \frown b \leq z \leq b,$$

respectively. In fact, these inequalities imply the equations

$$a \frown z = a \frown b = x \frown b$$

whereby, using the modularity, we get

$$x = x \frown (a \frown b) = x \frown (z \frown a) = (x \frown z) \frown a,$$

$$z = z \frown (a \frown b) = z \frown (x \frown b) = (z \frown x) \frown b.$$

3. Existence of minimal and maximal complements. For any elements u, v, a with $u \leq a \leq v$ let $R_u^v(a)$ denote the set of the relative complements of a in $[u, v]$. We prove the following

Theorem 2. *Let $[u, v]$ be an interval, satisfying the lower covering condition, of a lattice L . If a is an element of $[u, v]$ such that the length of interval $[u, a]$ is finite, then the length of the partly ordered set $R_u^v(a)$ is not greater than that of $[u, a]$.*

Corollary. *Every element of finite height of a complemented semimodular lattice has maximal as well as minimal complements.*

PROOF. Since the theorem is clearly true for the cases $a=u$ and $a=v$, we may restrict the discussion to the case when $u < a < v$.

Provided that the length of $[u, a]$ is n (where n is finite), there exists a chain⁴⁾

$$(6) \quad u = a_0 < a_1 < \dots < a_{n-1} < a_n = a$$

⁴⁾ $x < y$ means that x is covered by y .

between u and a . Let r denote any relative complement of a in $[u, v]$:

$$a \frown r = u, \quad a \smile r = v.$$

Form the chain

$$(7) \quad r = a_0 \smile r \leq a_1 \smile r \leq \dots \leq a_{n-1} \smile r \leq a_n \smile r = v.$$

We show that it is a maximal chain between r and v . Since

$$a_j \leq a_{j+1} \frown (a_j \smile r) \leq a_{j+1} \quad (j = 0, 1, \dots, n-1)$$

and $a_j < a_{j+1}$ by (6), either

$$a_j = a_{j+1} \frown (a_j \smile r) < a_{j+1}$$

or

$$a_{j+1} \frown (a_j \smile r) = a_{j+1}.$$

In the former case

$$a_j \smile r < a_{j+1} \frown (a_j \smile r) = a_{j+1} \smile r$$

by the lower covering condition. In the latter case $a_j \leq a_{j+1} \leq a_j \smile r$ whence $a_j \smile r \leq a_{j+1} \smile r \leq a_j \smile r$, i.e.

$$a_j \smile r = a_{j+1} \smile r.$$

Summing up the two cases we obtain

$$a_j \smile r \leq a_{j+1} \smile r \quad (j = 0, 1, \dots, n-1).$$

This means that the chain in (7) is a maximal one and its length is at most n . It follows ([3], p. 104) that *the length of the interval $[r, v]$ for any relative complement r of a in $[u, v]$ cannot be greater than n* . The length of $R_u^v(a)$ is, a fortiori, at most n . Thus the theorem is proved.

Finally we prove

Theorem 3. *Every element of a partition lattice has both minimal and maximal complements.*

PROOF. The customary proof of complementarity of partition lattices (see, e.g., [3], p. 148) proceeds, as is well-known, by giving effectively a complement to each element. It is easily seen that this complement is a minimal one. Thus existence of maximal complements is only to be shown.

Let A be an arbitrary element of a partition lattice P . In order to give a maximal complement M of A we begin by selecting one element from each A -class; the set of all selected elements will form a M -class. From the remaining elements of each A -class we select an element again; they will form another M -class. Then we continue constructing the M -classes in the same way (after we have selected all elements of an A -class, it will be left out of consideration at the construction of further M -classes). Clearly, M is a complement of A . Let C be any element of P greater than M . Then at least one of the C -classes is the set union of two or more M -classes and, therefore, it contains at least two elements belonging to the same A -class. Hence $A \frown C$ cannot be equal to the least element, formed by one-element classes, of P .

References

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