

## Ideal theory in the semiring $Z^+$

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It is well known that the ring of integers  $Z$  is a principal ideal ring and that  $Z$  is Noetherian. It is equally well known that the semiring (the definitions and results appearing in ALLEN [1] and [2] will be used throughout this paper) of non-negative integers  $Z^+$  is not a principal ideal semiring. It is generally assumed without question that  $Z^+$  is Noetherian; however, a proof has not been presented in the semiring literature, and prior to this paper, the ideals in  $Z^+$  have not been classified. It will be shown that there is only one basic type of ideal in  $Z^+$  and that all ideals in  $Z^+$  are related to this basic type in a natural way. Consequently, it will be an easy matter to classify ideals in  $Z^+$  and present a proof that  $Z^+$  is a Noetherian semiring. From these results, the discovery is made that  $Z^+$  is an "almost principal" ideal semiring.

When  $n \in Z^+$ , the notation  $T_n$  will be used to denote  $\{t \in Z^+ | t \geq n\} \cup \{0\}$ . The following are elementary facts concerning  $T_n$ .

**Theorem 1.** *If  $n \in Z^+$ , then  $T_n$  is an ideal in  $Z^+$  such that*

1.  $T_0 = T_1 = Z^+$ ,
2. *If  $1 \leq n < m$ , then  $T_m \subset T_n$  and  $T_m \neq T_n$ ,*
3.  $T_n \cup T_m = T_k$ , where  $k = \min \{n, m\}$ ,
4.  $T_n \cap T_m = T_q$ , where  $q = \max \{n, m\}$ , and
5.  $\bigcap \{T_i | i \in Z^+\} = \{0\}$ .

PROOF. Let  $a \in T_n$  and  $b \in T_n$ . Since  $a \geq n$  and  $b \geq n$ , it follows that  $a + b \geq 2n \geq n$ . Moreover, if  $k \in Z^+$ , where  $k \neq 0$ , then  $ka \geq kn \geq n$ . Therefore,  $a + b \in T_n$  and  $ka \in T_n$  and it is clear that  $T_n$  is an ideal in  $Z^+$ . The proofs of properties (1) through (5) are straightforward and will be omitted.

When  $a \in Z^+$  and  $b \in Z^+$ , the notation  $S(a, b)$  will be used to denote the set  $\{t \in Z^+ | a \leq t \leq b\}$ .

**Theorem 2.** *If  $n > 1$ , then  $S(n, 2n)$  is a finite basis for  $T_n$ .*

PROOF. Let  $p \in T_n$ . If  $p \in S(n, 2n)$  or  $p = cn$  for some  $c \in Z^+$ , then  $p$  is generated by  $S(n, 2n)$ . Let  $p > 2n$  and  $p \neq cn$  for any  $c \in Z^+$ . There exists a  $k \geq 2$  such that  $kn < p < (k+1)n$ . However, this guarantees the existence of an  $s < n$  such that  $kn + s = p$ , and it follows that  $n + s \in S(n, 2n)$ . Therefore,  $p = kn + s = (k-1)n + n + s$ , where  $n \in S(n, 2n)$  and  $(n+s) \in S(n, 2n)$ , and it follows that  $S(n, 2n)$  is a basis for  $T_n$ .

**Theorem 3.**  $Z^+$  satisfies the ascending chain condition on  $T_n$ -ideals.

PROOF. Let  $\{T_{n_\alpha}\}$  be an ascending chain of  $T_n$ -ideals in  $Z^+$ . It follows from Theorem 1 that  $\{n_\alpha\}$  is a nonincreasing sequence of positive integers. Since any nonincreasing sequence of positive integers is finite there exists  $\mu \in Z^+$  such that  $n_i = n_\mu$  for each  $i \geq \mu$ . Therefore  $T_{n_i} = T_{n_\mu}$  for each  $i \geq \mu$  and  $Z^+$  satisfies the ascending chain condition on  $T_n$ -ideals.

The following lemmas will be essential in the characterization of all ideals in  $Z^+$ . They also give some methods by which one can determine if an ideal in  $Z^+$  contains a  $T_n$ -ideal.

**Lemma 4.** Let  $I$  be an ideal in  $Z^+$ . If  $a \in I$ ,  $m \in Z^+$ , where  $m \neq 0$ , and  $S(ma, (m+1)a) \subset I$ , then there exists an  $n \in Z^+$  such that  $T_n \subset I$ .

PROOF. Suppose  $p \in Z^+$ ,  $p > (m+1)a$  and  $p \neq ca$  for  $c \in Z^+$ . Since there exists  $k \geq m+1$  such that  $ka < p < (k+1)a$ , one has  $ka + s = p$  for some  $s < a$ . Clearly  $s < a$  implies that  $ma + s \in S(ma, (m+1)a) \subset I$ . Therefore,  $p = ka + s = (k-m)a + ma + s \in I$ . Consequently,  $T_{ma} \subset I$  and the lemma follows.

**Lemma 5.** Let  $I$  be an ideal in  $Z^+$ . If there exists an  $a \in I$  such that  $a+1 \in I$ , then there exists an  $n$  such that  $T_n \subset I$ .

PROOF. If  $I$  is a  $T_n$ -ideal, the lemma is obvious. Suppose  $I$  is not a  $T_n$ -ideal and  $a$  is the least element in  $I$  such that  $a+1 \in I$ . Since  $I$  is an ideal, a series of simple calculations show that the following elements belong to  $I$ :

$$\begin{array}{l} a, a+1 \\ 2a, 2a+1, 2a+2 \\ 3a, 3a+1, 3a+2, 3a+3 \\ \dots\dots\dots \\ aa, aa+1, aa+2, aa+3, \dots, aa+a = a(a+1). \end{array}$$

The last row of elements is  $S(a^2, (a+1)a)$  and in view of Lemma 4, there exists an  $n \in Z^+$  such that  $T_n \subset I$ .

**Lemma 6.** Let  $a \in Z^+$  and  $b \in Z^+$  where  $a \neq 0$  and  $b \neq 0$ . If  $d$  is the greatest common divisor of  $a$  and  $b$ , then there exists  $s \in Z^+$  and  $t \in Z^+$  such that  $sa = tb + d$  or  $tb = sa + d$ .

PROOF. From elementary number theory, it is well known that  $d = s'a + t'b$  for some integers  $s'$  and  $t'$ . Since  $0 \leq d \leq a$ ,  $0 \leq d \leq b$  and both  $a$  and  $b$  are positive, it follows that  $s' \leq 0$  and  $t' \geq 0$ , or  $s' \geq 0$  and  $t' \leq 0$ . If perchance  $s' \leq 0$  and  $t' \geq 0$ , then  $tb = sa + d$  where  $0 \leq t' = t$  and  $0 \leq -s' = s$ . On the other hand, if  $s' \geq 0$  and  $t' \leq 0$ , then  $sa = tb + d$  where  $0 \leq s' = s$  and  $0 \leq -t' = t$ , and the conclusions follows.

The above lemma is necessary for the following:

**Lemma 7.** Let  $I$  be an ideal in  $Z^+$ ,  $a \in I$  and  $b \in I$ . If  $a$  and  $b$  are relatively prime, then there exists an  $n$  such that  $T_n \subset I$ .

PROOF. Since 1 is the greatest common divisor of  $a$  and  $b$ , the above lemma guarantees the existence of  $s \in Z^+$  and  $t \in Z^+$  such that  $sa = tb + 1$  or  $tb = sa + 1$ . Since  $I$  is an ideal it is clear that  $sa \in I$  and  $tb \in I$ . Consequently,  $sa + 1 \in I$  or  $tb + 1 \in I$  and the lemma follows from Lemma 5.

It is easy to see that for  $m \neq n$ ,  $T_m$  and  $T_n$  differ by at most a finite number of elements. Since  $Z^+ = T_1$ , it follows that  $Z^+$  differs from a  $T_n$ -ideal by at most a finite number of elements. Consequently, if  $I$  is an ideal in  $Z^+$  containing a  $T_n$ -ideal, then  $T_n \subset I \subset Z^+$  and it follows that  $Z^+$  and  $I$  differ by at most a finite number of elements. It will be shown that an ideal  $I$  in  $Z^+$  not containing a  $T_n$ -ideal differs from the multiples of some positive integer  $d > 1$  by at most a finite number of elements. Consequently, if  $I$  is an ideal in  $Z^+$  not containing a  $T_n$ -ideal, then there exist  $m \in Z^+$  and  $d \in Z^+$ , where  $d > 1$ , such that  $dT_m \subset I \subset (d)$ .

In view of the above remarks, the ideals in  $Z^+$  are classified according to the following definition.

**Definition 8.** An ideal  $I$  in  $Z^+$  will be called a  $T$ -ideal if  $T_k \subset I$  for some  $k \in Z^+$ . All other ideals in  $Z^+$  will be called  $M$ -ideals.

It is clear that  $Z^+$  is a  $T$ -ideal and  $\{0\}$  is an  $M$ -ideal. The following theorem gives a characterization of  $T$ -ideals in  $Z^+$  and will be used to show that  $Z^+$  is Noetherian.

**Theorem 9.** An ideal  $I$  in  $Z^+$  is a  $T$ -ideal if and only if  $I$  has a finite basis and  $I = K \cup T_k$  where  $T_k$  is the maximal  $T_n$ -ideal contained in  $I$  and  $K = \{t \in I \mid 0 < t < k\}$ .

PROOF. Suppose  $I$  is a  $T$ -ideal and  $T_n \subset I$ . Let  $S = \{n \in Z^+ \mid T_n \subset I\}$ . It is clear that  $S$  is a non-empty subset of  $Z^+$  and by the Well-Ordering Principle,  $S$  contains a least element, say  $k$ . By Theorem 1,  $T_n \subset T_k$  for each  $n \in S$  and it is clear that  $T_k$  is the maximal  $T_n$ -ideal contained in  $I$ . Letting  $K = \{t \in I \mid 0 < t < k\}$  one has  $I = K \cup T_k$ . According to Theorem 2,  $S(k, 2k)$  is a finite basis for  $T_k$ . Since  $K$  is a finite set,  $S(k, 2k) \cup K$  is a finite basis for  $I$ . The converse of the theorem is obvious.

**Theorem 10.** If  $n \in Z^+$  and  $d \in Z^+$ , then  $dT_n$  is an ideal in  $Z^+$  such that

1.  $dT_1 = (d)$  and  $dT_n = T_n$  if and only if  $d = 1$ ,
2.  $dT_n = \{0\}$  if and only if  $d = 0$ ,
3. If  $m < k$ , then  $dT_k \subset dT_m$ ,
4.  $dT_m \cup dT_k = dT_p$ , where  $p = \min \{m, k\}$ ,
5.  $dT_m \cap dT_k = dT_q$ , where  $q = \max \{m, k\}$ , and
6.  $\bigcap \{dT_n \mid n \in Z^+\} = \{0\}$ .

PROOF. Suppose  $x \in dT_n$  and  $y \in dT_n$ . Then there exist  $k \cong n$  and  $q \cong n$  such that  $x = kd$  and  $y = qd$ . Clearly,  $k + q \cong n$  and  $x + y = kd + qd = (k + q)d \in dT_n$ . If  $c \in Z^+$  where  $c \neq 0$ , then  $ck \cong n$  and  $cx = c(kd) = (ck)d \in dT_n$ . Therefore,  $dT_n$  is an ideal in  $Z^+$ . The proofs of (1) through (6) are straightforward and are omitted.

It will be shown that for any ideal  $I$  in  $Z^+$  there exist  $n \in Z^+$  and  $d \in Z^+$  such that  $dT_n$  is contained in  $I$ . Consequently, the  $dT_n$ -ideal is the basic type of ideal in  $Z^+$  and the study of ideals in  $Z^+$  is reduced to the problem of finding a maximal  $dT_n$ -ideal for each ideal in  $Z^+$ . It has already been observed in the previous theorem that  $dT_n = T_n$  if  $d = 1$  and  $dT_n = \{0\}$  if  $d = 0$ . Consequently, it only remains to study

the case for  $d > 1$ . For this purpose, in the remainder of this paper it will be assumed that  $d > 1$  unless otherwise stated.

The following three lemmas are analogues of well known properties of ideals in  $Z$ .

**Lemma 11.** *If  $p \in Z^+$ ,  $q \in Z^+$  and  $p$  divides  $q$ , then  $qT_n \subset pT_n$ .*

PROOF. Suppose  $a \in qT_n$ . There exists  $k \geq n$  such that  $a = kq$ . Since  $p$  divides  $q$ , there exists  $t \geq 1$  such that  $q = tp$ . Consequently,  $a = kq = k(tp) = (kt)p \in pT_n$ , since  $kt \geq n$ , and it follows that  $qT_n \subset pT_n$ .

**Lemma 12.** *If  $dT_c \subset bT_a$  then  $b$  divides  $d$ .*

PROOF. Suppose  $dT_c \subset bT_a$ . Since  $cd \in bT_a$ , there exists  $p \geq a$  such that  $cd = pb$  and  $b$  divides  $cd$ . By definition of  $dT_c$ ,  $(c+1)d \in bT_a$  and there exists  $q \geq a$  such that  $(c+1)d = qb$ . Consequently,  $b$  divides  $(c+1)d = cd + d$  and in view of the fact that  $b$  divides  $cd$ , one has  $b$  divides  $d$ .

**Lemma 13.** *If  $bT_a \cap dT_c \neq \{0\}$ , then there exist  $p \in Z^+$  and  $q \in Z^+$  such that  $qT_p \subset bT_a \cap dT_c$ .*

PROOF. Suppose  $x \in bT_a \cap dT_c$ . It is clear that  $xT_1 \subset bT_a$  and that  $xT_1 \subset dT_c$ , and the proof is complete.

The following two lemmas are essential to show that  $Z^+$  is Noetherian on  $dT_n$ -ideals.

**Lemma 14.** *Any ascending sequence  $\{bT_{a_j}\}$  is finite.*

PROOF. Suppose  $\{bT_{a_j}\}$  is an ascending sequence of ideals in  $Z^+$ . By Theorem 10,  $\{a_j\}$  is a decreasing sequence of positive integers and is therefore finite; i.e., there exists  $\alpha \in Z^+$  such that  $a_\alpha = a_n$  for each  $n \geq \alpha$ . Therefore,  $bT_{a_\alpha} = bT_{a_n}$  for each  $n \geq \alpha$ .

**Lemma 15.** *Any ascending sequence  $\{b_iT_a\}$  is finite.*

PROOF. Let  $\{b_iT_a\}$  be an ascending sequence of ideals in  $Z^+$ . In view of Lemma 12,  $b_i$  divides  $b$  for  $i \in \{2, 3, 4, \dots\}$ . Since  $b$  is finite and not zero there can only be a finite number of distinct  $b_i$ 's. Hence, there is an  $\alpha \in Z^+$  such that  $b_\alpha = b_n$  for each  $n \geq \alpha$  and it follows that  $b_\alpha T_a = b_n T_a$  for each  $n \geq \alpha$ .

**Theorem 16.**  *$Z^+$  satisfies the ascending chain condition on  $dT_n$ -ideals.*

PROOF. Let  $\{b_iT_{a_i}\}$  be an ascending chain of ideals in  $Z^+$ . By Lemma 15, there exists  $\alpha \in Z^+$  such that  $b_\alpha = b_i$  if  $i \geq \alpha$ . By Lemma 14, there exists  $\beta \in Z^+$  such that  $a_\beta = a_j$  if  $j \geq \beta$ . If  $k = \max\{\alpha, \beta\}$ , then  $b_kT_{a_k} = b_pT_{a_p}$  for  $p \geq k$ .

When  $x \in Z^+$ ,  $y \in Z^+$  and  $d \in Z^+$  where  $d > 1$ , denote by  $S_d(x, y)$  the set  $\{k \in Z^+ \mid x \leq k \leq y \text{ and } k = md \text{ for some } m \in \{Z^+\}\}$ .

**Lemma 17.**  *$S_d(nd, 2nd)$  is a finite basis for  $dT_n$ .*

PROOF. Let  $p = qd \in dT_n$ . If  $p \in S_d(nd, 2nd)$  or  $p = a(nd)$  for some  $a \in Z^+$ , then  $p$  is generated by  $S_d(nd, 2nd)$ . Suppose  $p \notin S_d(nd, 2nd)$  and  $p \neq a(nd)$  for  $a \in Z^+$ . Since  $p > 2nd$  and there exists  $k > 2$  such that  $knd < p < (k+1)nd$ , it follows that there exists an  $s = td < nd$  such that  $knd + s = p$ . However,  $s < nd$  implies that  $nd + s = nd + dt = nd + td = (n+t)d \in S_d(nd, 2nd)$ . Therefore,  $p = knd + s = (k-1)nd + nd + s$  and  $S_d(nd, 2nd)$  is a finite basis for  $dT_n$ .

**Lemma 18.** *Let  $I$  be an ideal in  $Z^+$ ,  $a \in I$  and  $d$  divide  $a$ , where  $d > 1$ . If there exists  $m \in Z^+$  such that  $S_d(ma, (m+1)a) \subset I$ , then there exists  $n \in Z^+$  such that  $dT_n \subset I$ .*

**PROOF.** If  $d$  divides  $a$ , then there exists  $b \in Z^+$  such that  $a = bd$  and it follows that  $S_d(ma, (m+1)a) = S_d((mb)d, (md+b)d)$ . Clearly, if  $p = qd$ , where  $mb \leq q \leq (m+1)b$ , then  $p \in S_d(ma, (m+1)a) \subset I$ . Suppose  $p = qd$ , where  $q > (m+1)b$ . It is clear that  $p > (m+1)a$  and there exists  $k \geq 1$  such that  $k(ma) \leq p \leq (k+1)ma$ . Consequently, there exists  $s \in Z^+$  such that  $s < ma$  and  $k(ma) + s = p$ . However,  $k(ma) + s = (kmb)d + s = p = qd$  and it follows that  $s = cd$  for some  $c \in Z^+$ . Moreover,  $ma + s < (m+1)a$  and it is easy to see that  $ma + s = m(bd) + cd = (mb + c)d \in S_d(ma, (m+1)a) \subset I$ . Therefore,  $ma + s \in I$  and  $(k-1)ma \in I$  together imply that  $p = qd = k(ma) + s = (k-1)ma + (ma + s) \in I$ . This shows that for each  $q \geq mb$ ,  $qd \in I$  and consequently, letting  $n = mb$  it is clear that  $dT_n \subset I$ .

**Theorem 19.** *Let  $I$  be an ideal in  $Z^+$ ,  $a \in I$  and  $b \in I$ . If  $a$  and  $b$  are relatively prime, then there exists  $n \in Z^+$  such that  $dT_n \subset I$ , where  $d$  is the greatest common divisor of  $a$  and  $b$ .*

**PROOF.** Since  $d$  is the greatest common divisor of  $a$  and  $b$ ,  $b = pd$  for some  $p \in Z^+$  and by Lemma 6, there exist positive integers  $s$  and  $t$  such that  $sa = tb + d$  or  $tb = sa + d$ . Since  $I$  is an ideal, it is clear that  $sa \in I$  and  $tb \in I$ . Consequently, if  $sa = tb + d$ , a series of simple calculations show that the following elements belong to  $I$ :

$$\begin{aligned} &bt, \quad bt + d, \\ &2bt, \quad 2bt + d, \quad 2bt + 2d, \\ &3bt, \quad 3bt + d, \quad 3bt + 2d, \quad 3bt + 3d \\ &\dots\dots\dots \\ &ptb, \quad ptb + d, \quad ptb + 2d, \quad ptb + 3d, \dots, \quad ptb + pd. \end{aligned}$$

Substituting  $b = pd$  in the last row one obtains

$$p^2td, (p^2t+1)d, (p^2t+2)d, \dots, (p^2t+p)d.$$

Since  $ptb + pd = pt + b = (ptb + 1)b$ , the last row is  $S_d(ptb, (pt+1)b)$  and Lemma 18 implies the existence of an  $n \in Z^+$  such that  $dT_n \subset I$ . On the other hand, if  $tb = sa + d$  a similar argument yields the same result.

The following theorem is needed for the characterization of  $M$ -ideals in  $Z^+$ .

**Theorem 20.** *If  $I$  is an  $M$ -ideal in  $Z^+$ , then there exist  $n \in Z^+$  and  $d \in Z^+$  such that  $dT_n \subset I$ .*

**PROOF.** If  $a \in I$  and  $b \in I$  where  $a$  and  $b$  are relatively prime, then Lemma 7 implies  $T_k \subset I$  for some  $k$ , a contradiction to the fact that  $I$  is an  $M$ -ideal. Consequently, if  $a \in I$  and  $b \in I$ , then their greatest common divisor, say  $d$ , is greater than 1, and the previous theorem gives the desired result.

The following theorem gives a structure and characterization of  $M$ -ideals in  $Z^+$  and is necessary to show that  $Z^+$  is Noetherian.

**Theorem 21.** An ideal  $I$  in  $Z^+$  is an  $M$ -ideal if and only if  $I$  has a finite basis and  $I = L \cup qT_p$ , where  $q > 1$ ,  $qT_p$  is a maximal  $dT_n$ -ideal contained in  $I$ , and  $L = \{t \in I \mid 0 < t < pq\}$ .

PROOF. The above theorem guarantees the existence of  $n$  and  $d > 1$  such that  $dT_n \subset I$ , if  $I$  is an  $M$ -ideal. Let  $S = \{d \in Z^+ \mid d \text{ is the greatest common divisor of some } a \in I \text{ and } b \in I\}$  and  $q$  be the least element in  $S$ . Theorem 19 assures that  $W = \{n \in Z^+ \mid qT_n \subset I\}$  is a non-empty subset of  $Z^+$ . Consequently, if  $p$  is the least element of  $W$ , then it is clear that  $qT_p \subset I$ . Suppose there exists  $bT_a \subset I$  such that  $qT_p \subset bT_a$ . It follows from Theorem 12 that  $b$  divides  $q$  and consequently  $b \leq q$ . Since  $b$  is the greatest common divisor of  $ba$  and  $b(a+1)$  one has  $b \in S$  and it follows that  $q \leq b$ . Consequently,  $b = q$ . By Theorem 10,  $a \leq p$  and since  $a \in W$  it follows that  $p \leq a$ . Consequently,  $p = a$  and  $qT_p = bT_a$ . Therefore  $qT_p$  is a maximal ideal in  $I$ . If  $c \in I$ ,  $c > pq$  and  $k$  is the greatest common divisor of  $c$  and  $pq$ , then  $c = ka$  for some  $a \in Z^+$ ,  $k \in S$  and it can be shown that  $q$  divides  $k$ . Thus there exists  $r \in Z^+$  such that  $k = rq$ . Consequently,  $pq < c = ka(rq)a = (ra)q$  and it follows that  $ra > p$  and  $c \in qT_p$ . If  $L = \{t \in I \mid 0 < t < pq\}$  then it is clear that  $I = L \cup qT_p$ . In view of Lemma 17,  $S_q(pq, 2pq)$  is a finite basis for  $I$ . The converse of the theorem is obvious.

Since any ideal in  $Z^+$  is either a  $T$ -ideal or an  $M$ -ideal Theorems 9 and 21 give a classification and structure for all ideals in  $Z^+$ . These results can now be used to obtain the following theorem.

**Theorem 22.**  $Z^+$  is a Noetherian semiring.

PROOF. In view of Theorem 9 and Theorem 21, any ideal in  $Z^+$  has a finite basis, and it follows that  $Z^+$  is Noetherian (see Allen [1]).

**Definition 23.** An ideal  $I$  in a semiring  $R$  will be called *almost principal* if there exists a finite set  $S \subset R$  such that  $I \cup S = P$ , where  $P$  is a principal ideal in  $R$ . The semiring  $R$  will be called an *almost principal ideal semiring* if every ideal in  $R$  is almost principal.

**Theorem 24.**  $Z^+$  is an almost principal ideal semiring.

PROOF. Let  $I$  be an ideal in  $Z^+$ . If  $I$  is a  $T$ -ideal, then by Theorem 9,  $I = K \cup T_n$ . Let  $S = \{t \in Z^+ \mid t \notin I\}$ . It is clear that  $S$  is a finite subset of  $Z^+$  and  $I \cup S = Z^+ = (1)$  is a principal ideal. If  $I$  is an  $M$ -ideal, then by Theorem 21,  $I = L \cup dT_n$ . Let  $S = \{td \mid t \in Z^+ \text{ and } td \notin I\}$ . It is clear that  $S$  is a finite subset of  $Z^+$  and  $I \cup S = (d)$  is a principal ideal. In either case  $I$  is an almost principal ideal and the result follows.

## References

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