

Generalisation of Gauss-Codazzi equations for Berwald's curvature tensor in a hypersurface of a Finsler space

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Introduction

In the present paper the authors wish to derive the generalisations of Gauss Codazzi equations by considering the induced Berwald's covariant derivative of a unit vector in the direction of the congruences of curves associated to a hypersurface of a Finsler space.

1. Notations and fundamental formulae

Let F_n be an n -dimensional Finsler space equipped with a positively homogeneous metric function $F(x, \dot{x})$. The fundamental metric tensor $g_{ij}(x, \dot{x})$ of F_n is defined by [1]¹⁾

$$(1.1) \quad g_{ij}(x, \dot{x}) = \frac{1}{2} \partial_{ij}^2 F^2(x, \dot{x}), \quad ^2)$$

where

$$(1.2) \quad C_{ijk}(x, \dot{x}) \stackrel{\text{def}}{=} \frac{1}{2} \partial_k g_{ij}(x, \dot{x}).$$

Let $X^i(x, \dot{x})$ be a vector field. The covariant derivatives of $X^i(x, \dot{x})$ with respect to x^k in the sense of Berwald and Cartan are given by

$$(1.4) \quad X_{(k)}^i = \partial_k X^i - (\partial_m^* X^i) G_{k\gamma}^m \dot{x}^\gamma + X^m G_{mk}^i$$

and

$$(1.5) \quad X_{|k}^i = \partial_k X^i - (\partial_m^* X^i) \Gamma_{\gamma k}^{*m} \dot{x}^\gamma + X^m \Gamma_{mk}^{*i}$$

¹⁾ Numbers in brackets refer to the references given at the end of the paper.

²⁾ $\partial_i^* = \partial/\partial \dot{x}^i$, $\partial_i = \partial/\partial x^i$ and $\partial_{ij}^2 = \partial^2/\partial \dot{x}^i \partial \dot{x}^j$.

respectively, where $G_{mk}^i(x, \dot{x})$ and $\Gamma_{mk}^{*i}(x, \dot{x})$ are connection parameters symmetric in its lower indices and are related by the equation

$$(1.6) \quad \Gamma_{\gamma k}^{*i} \dot{x}^\gamma = G_{\gamma k}^i \dot{x}^\gamma.$$

We shall now consider a hypersurface F_{n-1} of F_n represented by

$$(1.7) \quad x^i = x^i(u^\alpha), \quad (\alpha = 1, 2, \dots, n-1)$$

where u^α are the parameter. The matrix $\|B_\alpha^i\|$ of the projection parameters has rank $n-1$. We shall use the following notations

$$(1.8) \quad B_\alpha^i = \partial_\alpha x^i, \quad B_{\alpha\beta}^i = \partial_{\alpha\beta}^2 x^i, \quad B_{\alpha\beta \dots \gamma}^{ij \dots k} = B_\alpha^i B_\beta^j \dots B_\gamma^k.$$

The hypersurface vector $\dot{u}^\alpha$ and a vector $\dot{x}^i$ of F_n are related by

$$(1.9) \quad \dot{x}^i = \dot{u}^\alpha B_\alpha^i.$$

The metric (induced) tensor $g_{\alpha\beta}(u, \dot{u})$ of F_{n-1} is given by

$$(1.10) \quad g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x}) B_\alpha^{ij} B_\beta^{ij}.$$

$g^{\alpha\beta}(u, \dot{u})$ are the inverse of the tensor $g_{\alpha\beta}(u, \dot{u})$; therefore we can construct the following entities:

$$(1.11) \quad B_i^\alpha = g^{\alpha\epsilon} B_\epsilon^j g_{ij}$$

which satisfies the identity

$$(1.12) \quad B_i^\alpha B_\beta^i = \delta_\beta^\alpha = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

At each point P of F_{n-1} we can define the unit normal vectors $N^i(x, \dot{x})$ with respect to the tangential direction $\dot{x}^i$ at P by the set of equations.

$$(1.13) \quad \text{a) } g_{ij} N^j B_\alpha^i = N_i B_\alpha^i = 0 \quad \text{b) } g^{ij} N_i N_j = N^i N_i = 1$$

which imply

$$(1.14) \quad \text{a) } N^i B_i^\alpha = 0, \quad \text{b) } g_{ij} N^i N^j = 1.$$

The induced symmetric tensor $A_{\beta\gamma}^\alpha$ and the connection parameters $G_{\beta\gamma}^\alpha$ (Sinha and Singh [3]) of F_{n-1} are given by

$$(1.15) \quad A_{\beta\gamma}^\alpha(u, \dot{u}) = A_{hk}^i(x, \dot{x}) B_i^\alpha B_\beta^{hk} B_\gamma^{hk}$$

and

$$(1.16) \quad G_{\beta\gamma}^\alpha = B_i^\alpha \{B_{\beta\gamma}^i + B_{\beta\gamma}^{hk} (G_{hk}^i - C_{hk|m}^i \dot{x}^m)\} + A_{\beta\gamma}^\alpha + C_{\beta\gamma|q}^\alpha \dot{u}^q,$$

where the quantity $A_{\beta\gamma}^\alpha = g^{\alpha\epsilon} A_{\beta\epsilon\gamma}$ satisfies the identities

$$A_{\beta\gamma}^\alpha \dot{u}^\beta = N M_\gamma^\alpha \quad \text{and} \quad A_{\alpha\beta}^\epsilon \dot{u}^\alpha \dot{u}^\beta = 0.$$

With the help of the above quantities we can define an induced mixed derivative denoted by $T_{\alpha((\beta))}^i$ in the sense of Berwald as follows

$$(1.17) \quad T_{\alpha((\beta))}^i = \partial_\beta T_\alpha^i - (\partial_\gamma T_\alpha^i) G_\beta^\gamma + T_\alpha^s G_{sh}^i B_\beta^h - T_\epsilon^i G_{\alpha\beta}^\epsilon.$$

With the help of the above equation, we can define the mixed tensor $V_{\alpha\beta}^i$ as

$$(1.18) \quad B_{\alpha((\beta))}^i \stackrel{\text{def}}{=} V_{\alpha\beta}^i = B_{\alpha\beta}^i - B_{\alpha}^i G_{\alpha\beta}^e + G_{hk}^i B_{\alpha\beta}^{hk}$$

which are regarded as vectors of the imbedding space F_{n-1} and are normal to F_{n-1} . We may write $V_{\alpha\beta}^i$, [3], as

$$(1.19) \quad V_{\alpha\beta}^i = N^i \tilde{\Omega}_{\alpha\beta} - B_{\alpha}^i T_{\alpha\beta}^e + C_{hk|m}^i \dot{x}^m B_{\alpha\beta}^{hk},$$

where $\tilde{\Omega}_{\alpha\beta}$ is the second fundamental tensor symmetric in its lower indices and

$$(1.20) \quad T_{\alpha\beta}^e \stackrel{\text{def}}{=} A_{\alpha\beta}^e + C_{\alpha\beta|q}^e \dot{u}^q.$$

The induced derivative $N_{((\beta))}^i$ of type (1.17) is given by (Sinha and Singh [3]):

$$(1.21) \quad N_{((\beta))}^i = -\tilde{\Omega}_{\alpha\beta} g^{\alpha\delta} B_{\delta}^i + E_m^i V_{\alpha\beta}^m \dot{u}^e + N^h C_{hk|m}^i \dot{x}^m B_{\beta}^k,$$

where

$$(1.22) \quad E_m^i = M_m N^i - 2M_m^i$$

and

$$(1.23) \quad M_l^i = C_{lp}^i N^p, \quad M_l = C_{pkl} N^p N^k = M_{kl} N^k.$$

2. Generalised Gauss Codazzi equations

Consider a set of congruences of curves such that one curve of each of them passes through every point of F_{n-1} . We consider the contravariant components of a unit vector in the direction of a curve of a congruence of curves as a linear combination of tangent vector B_{α}^i and normal vector N^i as

$$(2.1) \quad \lambda^i = t^{\alpha} B_{\alpha}^i + dN^i,$$

where t^{α} and d are parameters. Taking the mixed derivative of λ^i with respect to u^{β} of type (1.17) we get

$$(2.2) \quad \lambda_{((\beta))}^i = B_{\alpha}^i t_{((\beta))}^{\alpha} + t^{\alpha} B_{\alpha((\beta))}^i + N^i d_{((\beta))} + dN_{((\beta))}^i,$$

again taking the covariant derivative of (2.1) with respect to u^{γ} of type (1.17) and subtracting the equation obtained by interchanging the indices β and γ we get

$$(2.3) \quad \lambda_{[(\beta))((\gamma))]}^i = B_{\alpha}^i t_{[(\beta))((\gamma))]}^{\alpha} + t^{\alpha} B_{\alpha[(\beta))((\gamma))]}^i + dN_{[(\beta))((\gamma))]}^i + N^i d_{[(\beta))((\gamma))]}.$$

With the help of equation (1.18) we can write

$$(2.4) \quad B_{\alpha[(\beta))((\gamma))]}^i = V_{\alpha[\beta((\gamma))]}^i. \quad ^3)$$

Thus by substituting equations (1.19) and (1.21) in (2.4) we obtain

$$(2.5) \quad \begin{aligned} B_{\alpha[(\beta))((\gamma))]}^i &= B_{\delta}^i \{T_{\alpha[\beta}^e T_{\gamma]}^{\delta} - \tilde{\Omega}_{\alpha[\beta} \tilde{\Omega}_{\gamma]}^{\delta} g^{\epsilon\delta} - T_{\alpha[\beta((\gamma))]}^{\delta}\} + \\ &+ N^i \{\tilde{\Omega}_{\alpha[\beta((\gamma))]} - T_{\alpha[\beta}^{\delta} \Omega_{\gamma]}^{\delta}\} + C_{hk|m}^i \dot{x}^m \{\tilde{\Omega}_{\alpha[\beta} B_{\gamma]}^k N^h - T_{\alpha[\beta}^{\delta} B_{\gamma]}^{hk}\} + \\ &+ E_m^i V_{\alpha[\gamma}^m \tilde{\Omega}_{\beta]}^{\delta} \dot{u}^q + (C_{hk|m}^i \dot{x}^m B_{\alpha[\beta)}^{hk} B_{\gamma]}^m). \quad ^4) \end{aligned}$$

³⁾ $2x_{[\alpha\beta]} = x_{\alpha\beta} - x_{\beta\alpha}$.

⁴⁾ Indices in brackets $\langle \rangle$ are free from symmetric and skew symmetric parts.

Similarly, by using equation (1.19) and (1.21) we get

$$(2.6) \quad \begin{aligned} N_{[(\beta))((\gamma))]}^i &= B_{\delta}^i \{T_{\varepsilon[\gamma]}^{\delta} \tilde{Q}_{\beta]}^{\varepsilon} g^{x\varepsilon} - (g^{x\delta} \tilde{Q}_{\alpha[\beta]}^{\delta})_{((\gamma))]} \} + \\ &+ E_m^i (V_{\varepsilon[\beta]}^m \dot{u}^{\varepsilon})_{((\gamma))]} + E_m^i (V_{\beta]}^m \dot{u}^{\varepsilon})_{((\gamma))]} + (C_{hk|m}^i \dot{x}^m)_{[(\gamma))]} B_{\beta]}^k N^h + \\ &+ C_{hk|m}^i \dot{x}^m \{ (E_s^h V_{\varepsilon[\gamma]}^s \dot{u}^{\varepsilon} + N^l C_{ls|a}^h \dot{x}^a B_{[\gamma]}^s) B_{\beta]}^k \}. \end{aligned}$$

We have $\lambda_{((\beta))}^i = \lambda_{(h)}^i B_{\beta}^h$ and $\lambda_{((\beta))((\gamma))}^i = \lambda_{(h)(k)}^i B_{\beta\gamma}^{hk} + \lambda_{(h)}^i V_{\beta\gamma}^h$.

Thus we have

$$(2.7) \quad \lambda_{[(\beta))((\gamma))]}^i = \lambda_{[(h)(k)]}^i B_{\beta\gamma}^{hk}.$$

But we know the following commutation formula [1]

$$(2.8) \quad 2\lambda_{[(h)(k)]}^i = \lambda^j H_{jhk}^i - (\partial_j \lambda^i) H_{hk}^j.$$

From equations (2.7) and (2.8), we get

$$(2.9) \quad 2\lambda_{[(\beta))((\gamma))]}^i = (\lambda^j H_{jhk}^i - \partial_j \lambda^i H_{hk}^j) B_{\beta\gamma}^{hk}.$$

Similarly for t^{δ} we have

$$(2.10) \quad 2t_{[(\beta))((\gamma))]}^{\delta} = t^{\varepsilon} H_{\varepsilon\beta\gamma}^{\delta} - (\partial_{\varepsilon} t^{\delta}) H_{\beta\gamma}^{\varepsilon},$$

where

$$(2.11) \quad H_{\varepsilon\beta\gamma}^{\delta}(u, \dot{u}) = 2 \{ \partial_{[\gamma} G_{\beta]}^{\delta} + G_{\varepsilon[\beta}^{\delta} G_{\gamma]}^{\delta} + G_{\varepsilon\beta}^{\delta} G_{\gamma]}^{\delta} \}$$

and

$$(2.12) \quad H_{\beta\gamma}^{\varepsilon}(u, \dot{u}) = 2 \{ \partial_{[\gamma} \partial_{\beta]} G^{\varepsilon} + G_{\varepsilon[\gamma}^{\varepsilon} G_{\beta]}^{\varepsilon} \}.$$

The induced covariant derivative of d in the sense of Berwald is given by

$$(2.13) \quad d_{((\beta))} = \partial_{\beta} d - (\dot{\partial}_{\gamma} d) G_{\beta}^{\gamma}.$$

Therefore, we get

$$(2.14) \quad d_{[(\beta))((\gamma))]} = \partial_{\varepsilon} d \{ \partial_{[\beta} G_{\gamma]}^{\varepsilon} + G_{\varepsilon[\beta}^{\varepsilon} G_{\gamma]}^{\varepsilon} \}.$$

Substituting equations (2.5), (2.6), (2.9), (2.10) and (2.14) in (2.3) we obtain

$$(2.15) \quad \begin{aligned} \lambda^j H_{jhk}^i B_{\beta\gamma}^{hk} &= H_{\varepsilon\beta\gamma}^{\delta} t^{\varepsilon} B_{\delta}^i + (\partial_{\alpha} \lambda^i) H_{hk}^{\alpha} B_{\beta\gamma}^{hk} - B_{\delta}^i (\partial_{\varepsilon} t^{\delta}) H_{\beta\gamma}^{\varepsilon} + \\ &+ 2 [B_{\delta}^i \{ t^{\alpha} (T_{\alpha[\beta}^{\varepsilon} T_{\gamma]}^{\delta} - \tilde{Q}_{\alpha[\beta}^{\varepsilon} \tilde{Q}_{\gamma]}^{\delta}) g^{x\varepsilon} - T_{\alpha[\beta}^{\delta} \tilde{Q}_{\gamma]}^{\delta} \} + d (T_{\varepsilon[\gamma}^{\delta} \tilde{Q}_{\beta]}^{\varepsilon} g^{x\varepsilon} + (g^{x\delta} \tilde{Q}_{\alpha[\beta]}^{\delta})_{((\gamma))]} \}) + \\ &+ N^i \{ t^{\alpha} (\tilde{Q}_{\alpha[\beta}^{\delta} \tilde{Q}_{\gamma]}^{\delta} - T_{\alpha[\beta}^{\delta} \tilde{Q}_{\gamma]}^{\delta}) + \partial_{\varepsilon} d (\partial_{[\beta} G_{\gamma]}^{\varepsilon} + G_{\varepsilon[\beta}^{\varepsilon} G_{\gamma]}^{\varepsilon}) \} + \\ &+ E_m^i \{ t^{\alpha} V_{\varepsilon[\gamma}^m \tilde{Q}_{\beta]}^{\varepsilon} \dot{u}^{\varepsilon} + d (V_{\varepsilon[\beta}^m \dot{u}^{\varepsilon})_{((\gamma))]} \} + C_{hk|m}^i \dot{x}^m \{ t^{\alpha} (\tilde{Q}_{\alpha[\beta}^{\delta} B_{\gamma]}^k N^h - T_{\alpha[\beta}^{\delta} B_{\gamma]}^{hk}) + \\ &+ d (E_s^h V_{\varepsilon[\gamma]}^s \dot{u}^{\varepsilon} + N^e C_{\varepsilon s|a}^h \dot{x}^a B_{[\gamma]}^s) B_{\beta]}^k \} + t^{\alpha} (C_{hk|m}^i \dot{x}^m B_{\alpha[\beta]}^{hk})_{((\gamma))]} + \\ &+ d \{ E_m^i (V_{\varepsilon[\gamma]}^m \dot{u}^{\varepsilon} + (C_{hk|m}^i \dot{x}^m)_{((\gamma))]} B_{\beta]}^k N^h \}]. \end{aligned}$$

Multiplying equation (2.15) by B_i^δ and using equations (1.12), (1.14) and (1.22) we obtain

$$\begin{aligned}
 \lambda^n H_{nhk}^i B_{\beta\gamma}^{hk} B_i^\delta &= H_{\varepsilon\beta\gamma}^\delta t^\varepsilon + B_i^\delta (\partial_n^\varepsilon \lambda^i) H_{hk}^n B_{\beta\gamma}^{hk} - (\partial_\varepsilon^\delta t^\delta) H_{\beta\gamma}^\varepsilon + \\
 &+ 2[t^\alpha \{ (T_{\alpha[\beta}^\varepsilon T_{\varepsilon\gamma]}^\delta - \tilde{Q}_{\alpha[\beta}^\varepsilon \tilde{Q}_{\gamma]}^\delta) g^{\varepsilon\delta} - T_{\alpha[\beta}^\delta \tilde{Q}_{\gamma]}^\delta \} + d(T_{\varepsilon[\gamma}^\delta \tilde{Q}_{\beta]}^\delta) g^{\varepsilon\delta} + (g^{\alpha\delta} \tilde{Q}_{\alpha[\beta}^\delta)_{((\gamma))}] \} + \\
 (2.16) \quad &+ B_i^\delta \{ C_{hk|m}^i \dot{x}^m (t^\alpha (\tilde{Q}_{\alpha[\beta}^\delta B_{\gamma]}^k N^h - T_{\alpha[\beta}^\delta B_{\gamma]}^{hk}) + d(E_S^h V_{\varepsilon[\gamma}^S \dot{u}^\varepsilon + N^j C_{jS|a}^h \dot{x}^a B_{[\gamma]}^S) B_{\beta]}^k) - \\
 &- 2M_m^i (t^\alpha V_{\varepsilon[\gamma}^m \tilde{Q}_{\beta]}^\delta \dot{u}^\varepsilon + d(V_{\varepsilon[\beta}^m \dot{u}^\varepsilon)_{((\gamma))}) + t^\alpha (C_{hk|m}^i \dot{x}^m B_{\alpha[\beta}^{hk})_{((\gamma))}) + \\
 &+ d(E_{m[[(\gamma))] }^i V_{\beta]}^m \dot{u}^\varepsilon + (C_{hk|m}^i \dot{x}^m)_{[(\gamma))] } B_{\beta]}^k N^h \} \}.
 \end{aligned}$$

Again multiplying (2.15) by N_i and using equation (1.13) we get

$$\begin{aligned}
 N_i \lambda^n H_{nhk}^i B_{\beta\gamma}^{hk} &= (\partial_n^\varepsilon \lambda^i) N_i H_{hk}^n B_{\beta\gamma}^{hk} + 2[t^\alpha (\tilde{Q}_{\alpha[\beta}^\delta \tilde{Q}_{\gamma]}^\delta) - T_{\alpha[\beta}^\delta \tilde{Q}_{\gamma]}^\delta] + \\
 &+ \partial_\varepsilon^\delta d\{ \partial_{[\beta} G_{\gamma]}^\varepsilon + G_{\varepsilon[\beta}^\delta G_{\gamma]}^\varepsilon \} + N_i \{ E_m^i (t^\alpha V_{\varepsilon[\gamma}^m \tilde{Q}_{\beta]}^\delta \dot{u}^\varepsilon + d(V_{\varepsilon[\beta}^m \dot{u}^\varepsilon)_{((\gamma))}) + \\
 (2.17) \quad &+ C_{hk|m}^i \dot{x}^m (t^\alpha (\tilde{Q}_{\alpha[\beta}^\delta B_{\gamma]}^k N^h - T_{\alpha[\beta}^\delta B_{\gamma]}^{hk}) + d(E_S^h V_{\varepsilon[\gamma}^S \dot{u}^\varepsilon + N^j C_{jS|a}^h \dot{x}^a B_{[\gamma]}^S) B_{\beta]}^k) + \\
 &+ t^\alpha (C_{hk|m}^i \dot{x}^m B_{\alpha[\beta}^{hk})_{((\gamma))}) + d(E_{m[[(\gamma))] }^i V_{\beta]}^m \dot{u}^\varepsilon + (C_{hk|m}^i \dot{x}^m)_{[(\gamma))] } B_{\beta]}^k N^h \} \}.
 \end{aligned}$$

Equations (2.16) and (2.17) can be regarded as generalisation of Gauss Codazzi equation in a hypersurface F_{n-1} imbedded in a Finsler space F_n .

3. Particular cases

We can choose a congruence of curves in three different ways. Firstly it is supposed to be normal to F_{n-1} i.e. $\lambda^i = dN^i$. Secondly it is such that the vector with the contravariant components λ^i in the direction of the curve of the congruence is only depending upon the tangent vectors B_α^i i.e. $\lambda^i = t^\alpha B_\alpha^i$; thirdly it can be tangential to the curve $\dot{x}^i = \dot{u}^\alpha B_\alpha^i$.

References

- [1] H. RUND, The differential geometry of Finsler spaces, *Berlin*, 1959.
- [2] H. RUND, Curvature properties of hypersurfaces of Finsler spaces and Minkowskian spaces, *Tensor, N. S.*, **14** (1963), 226—244.
- [3] B. B. SINHA and S. P. SINGH, A generalisation of Gauss Codazzi equation for the Berwald's curvature tensor, *Tensor, N. S.* **22** (1971), 112—116.
- [4] B. B. SINHA and S. P. SINGH, A generalisation of Gauss and Codazzi equations for the first curvature of Cartan. *Progress of Math.* **6** (1972), 1—6.

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