

Characterizations of the Baer radical class by almost nilpotent rings

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To the memory of my beloved friend Prof. A. Kertész

All rings considered, are associative. A radical class will always mean a radical class in the sense of Kurosh and Amitsur. For the terminology and basic results we refer to [1] and [3]. In [2] VAN LEEUWEN and HEYMAN introduced and investigated almost nilpotent rings. A ring A is called an *almost nilpotent* one, if every proper homomorphic image of A is nilpotent.*) It has been proved in [2] that an almost nilpotent ring is either a Baer radical ring or Baer semisimple. In this note we generalize this statement and give two characterizations of the class of Baer (lower) radical rings (Corollaries 1 and 2).**)

In what follows $\mathbf{L}$ and $\mathbf{O}$ will denote the class of all almost nilpotent rings and that of one element rings, respectively. Let us recall that the class $\mathbf{B}$ of all Baer radical rings is just the lower radical class $\mathcal{LZ}$ of the class $\mathbf{Z}$ of all zero rings, further $\mathbf{B}$ is the upper radical class $\mathcal{UP}$ of the class $\mathbf{P}$ of all prime rings. For any radical class $\mathbf{R}$ the class of all $\mathbf{R}$ -semisimple rings will be denoted by $\mathcal{SR}$.

Let $\mathbf{R}$ be a radical class and consider the following condition:

(L) *Any almost nilpotent ring A is either $\mathbf{R}$ -radical or $\mathbf{R}$ -semisimple.*

Theorem 1. *Let $\mathbf{R}$ be a radical class such that $\mathbf{R} \cap \mathbf{L} \neq \mathbf{O}$. $\mathbf{R}$ satisfies condition (L) iff $\mathbf{R}$ contains the class $\mathbf{Z}$ of all zero rings.*

PROOF. Assume $\mathbf{Z} \subseteq \mathbf{R}$. In this case the class $\mathbf{R}$ contains all nilpotent rings, too. Let A be an almost nilpotent ring. If I is an ideal of A such that $0 \neq I \neq A$, then A/I is nilpotent and so I is not a prime ideal of A . Hence either A has no prime ideals or only 0 is a prime ideal of A . In the first case we have $A \in \mathbf{B} = \mathcal{LZ} \subseteq \mathbf{R}$. In the second case $A \in \mathcal{SB}$ holds, and we have $A/\mathbf{R}(A) \in \mathcal{SR} \subseteq \mathcal{SB}$. Hence either $A \in \mathbf{R}$ or $A/\mathbf{R}(A)$ is not nilpotent. In the latter case since $A \in \mathbf{L}$, it follows $\mathbf{R}(A) = 0$ i.e. $A \in \mathcal{SR}$. Thus condition (L) is established.

Conversely, suppose that the radical class $\mathbf{R}$ does not contain all zero rings. Then there exists a zero ring Z such that $Y = Z/\mathbf{R}(Z) \neq 0$, moreover, $Y \in \mathcal{SR}$. Since every semisimple class is hereditary and Y is a zero ring, so $\mathcal{SR}$ contains a zero

*) We mean that each simple ring is almost nilpotent.

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ring $Z(n)$ over the cyclic group $C(n)$ for some $n=2, 3, \dots, \infty$. If $\mathbf{R}$ contains a zero ring $X \neq 0$, then the direct sum $A = X \oplus Z(n)$ is almost nilpotent (in fact a zero ring) such that $0 \neq X = \mathbf{R}(A) \neq A$ and condition (L) is not satisfied. If $\mathbf{R} \cap \mathbf{Z} = \mathbf{O}$, then clearly $\mathcal{S}\mathbf{R}$ contains all nilpotent rings. Since $\mathbf{R} \cap \mathbf{L} \neq \mathbf{O}$, there exists a ring $A \neq 0$ in $\mathbf{R} \cap \mathbf{L}$. A can be decomposed into a subdirect sum $A \cong \sum'_{\alpha} A_{\alpha}$ of subdirectly irreducible rings A_{α} . Taking into account that A is almost nilpotent, either each A_{α} is nilpotent or $A \cong A_{\alpha}$ is subdirectly irreducible. The first case is not possible because $\mathbf{R}$ is homomorphically closed and $\mathcal{S}\mathbf{R}$ contains all nilpotent rings. Hence A is subdirectly irreducible. Since $\mathcal{S}\mathbf{R}$ contains all nilpotent rings, by $0 \neq A \in \mathbf{R}$ it follows that A is not nilpotent. Denoting the heart of A by H and taking into account $A \in \mathbf{L}$, we have $0 \neq (A^n)^2 \subseteq H^2$. Consequently H is an idempotent simple ring. We exhibit $H \in \mathbf{R}$. In the case $H \in \mathcal{S}\mathbf{R}$ being A/H nilpotent, we get $A/H \in \mathcal{S}\mathbf{R}$. Taking into account that every semisimple class is closed under forming extensions, it follows $A \in \mathcal{S}\mathbf{R}$ contradicting $0 \neq A \in \mathbf{R}$. Hence $H \in \mathbf{R}$ holds. Consider the ring B defined as follows:

$$B^+ = H^+ \oplus C^+$$

where C^+ is the cyclic additive group $C(p)$ if H has characteristic p and C^+ denotes the additive group Q^+ of rationals if H has characteristic 0. Further, the product of any two elements (k_1, c_1) and (k_2, c_2) of B should be defined by

$$(k_1, c_1)(k_2, c_2) = (k_1k_2 + c_2k_1 + c_1k_2, 0).$$

Then B is a ring, moreover, B is obviously almost nilpotent and $0 \neq H = \mathbf{R}(B) \neq B$. Thus the almost nilpotent ring B does not satisfy condition (L).

Corollary 1. *A radical class $\mathbf{R}$ coincides with the class $\mathbf{B}$ of Baer radical rings iff*

- (i) $\mathbf{R}$ satisfies condition (L);
- (ii) $\mathbf{R} \cap \mathbf{L} \neq \mathbf{O}$;
- (iii) for any radical class $\mathbf{R}'$ with properties (i) and (ii) the inclusion $\mathbf{R} \subseteq \mathbf{R}'$ holds.

The assertion is obvious in view of Theorem 1 and $\mathbf{B} = \mathcal{L}\mathbf{Z}$.

The next example shows that condition (ii) cannot be omitted from Corollary 1. Let V be a vector space of dimension $\aleph_{\omega_0}$ over a division ring D and let W denote the ring of all linear transformations of V . As it is known (cf. Example 11 of [1] on pp. 109–110) the proper ideals of W are the following ones:

$$I_n = \{A \in W \mid \dim \operatorname{Im} A < \aleph_n\}$$

for $n=0, 1, 2, \dots, \omega_0$. Consider the class $\mathbf{W}$ of all homomorphic images of the ring W . Obviously $\mathbf{W}$ consists of rings with unity and none of them is simple. Take the lower radical class $\mathcal{L}\mathbf{W}$ determined by the class $\mathbf{W}$. Taking into consideration that the class $\mathbf{L}$ of almost nilpotent rings is hereditary (cf. [2] Lemma) and that the semi-simple class $\mathcal{S}\mathcal{L}\mathbf{W}$ contains all nilpotent rings as well as simple rings, it follows $\mathbf{L} \subseteq \mathcal{S}\mathcal{L}\mathbf{W}$. Thus $\mathcal{L}\mathbf{W} \cap \mathbf{L} = \mathbf{O}$ though $\mathcal{L}\mathbf{W} \neq \mathbf{O}$ and $\mathcal{L}\mathbf{W}$ satisfies condition (L).

Confining ourselves to hereditary radicals, condition (ii) can be eliminated from Corollary 1.

Theorem 2. *A hereditary radical class $\mathbf{R} \neq \mathbf{O}$ satisfies condition (L) iff $\mathbf{Z} \subseteq \mathbf{R}$.*

PROOF. By Theorem 1 $\mathbf{Z} \subseteq \mathbf{R}$ implies that $\mathbf{R}$ satisfies condition (L). Suppose $\mathbf{Z} \not\subseteq \mathbf{R}$. If $\mathbf{R}$ contains a zero ring $X \neq 0$ then, as we have already seen in the proof of Theorem 1, condition (L) is not fulfilled. If all zero rings and so all nilpotent rings are contained in $\mathcal{S}\mathbf{R}$ then, taking into consideration $\mathbf{R} \neq \mathbf{O}$ and that $\mathcal{S}\mathbf{R}$ is closed under forming subdirect sums, $\mathcal{S}\mathbf{R}$ does not contain all subdirectly irreducible rings. Hence there exists a subdirectly irreducible ring A such that for the heart H of A the relation $0 \neq H \in \mathbf{R}(A)$ holds. Since $\mathbf{R}$ is hereditary, it follows $H \in \mathbf{R}$. Take the direct sum $B = H \oplus Z(p)$ where $Z(p)$ is the cyclic zero ring of prime order p . B is neither in $\mathbf{R}$ nor in $\mathcal{S}\mathbf{R}$, though B is almost nilpotent. Thus condition (L) is not satisfied.

Corollary 2. A hereditary radical class $\mathbf{R}$ coincides with the class $\mathbf{B}$ of Baer radical rings iff

- (i) $\mathbf{R}$ satisfies condition (L)
- (ii) for any hereditary radical class $\mathbf{R}'$ satisfying condition (L) the relation $\mathbf{R} \subseteq \mathbf{R}'$ holds.

Corollary 3. A radical class $\mathbf{R} \neq \mathbf{O}$ satisfying condition (L) but not containing the class $\mathbf{Z}$, is never hereditary.

References

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