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Remark on Ky Fan convexity

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Abstract. In the paper is proved that the Nikaido-Isolda's theorem fails to hold if concavity is replaced by Ky Fan concavity.

Let X, Y be arbitrary sets. The function

$$f : X \times Y \to \mathbb{R}$$

is called Ky Fan concave in the variable x if

$$\forall x_1, x_2 \in X \quad \forall \lambda \in [0,1] \quad \exists x_3 \in X \quad \forall y.$$
$$f(x_3, y) \geq \lambda f(x_1, y) + (1 - \lambda) f(x_2, y).$$

The concavity with respect to y is defined symmetrically. In [1] the authors stated the following

Theorem. *There exists functions $f_1, f_2 \in C^\infty([0,1] \times [0,1])$ such that f_1 is Ky Fan concave in the variable x, f_2 is Ky Fan concave in the variable y and the pair f_1, f_2 has no saddle point i.e. there is no point (x_0, y_0) satisfying*

$$f_1(x_0, y_0) \geq f_1(x_1 y_0) \quad \forall x$$
$$f_2(x_0, y_0) \geq f_2(x_0 y) \quad \forall y.$$

This is a counterexample showing that the Nikadio-Isolda theorem fails to hold if concavity is replaced by Ky Fan concavity. The proof given in [1] was not correct; it suggested that there are polynomials f_1, f_2

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satisfying Theorem. In fact we do not know whether there are analytical functions f_1, f_2 satisfying Theorem.

Proof of Theorem. Define the functions $k_1, k_2 : [0,1] \times [0,1]$ as follows. Let $0 < \delta < \frac{1}{4}$ be fixed. For $0 \le x \le \frac{1}{4}$ the function $k_1((x), k_2(x))$ varies linearly from $(0,1)$ to $\left(\frac{1}{2}+\delta, \frac{1}{2}-\delta\right)$; for $\frac{1}{4} \le x \le \frac{1}{2}$ it goes linearly from $\left(\frac{1}{2}+\delta, \frac{1}{2}-\delta\right)$ to $(0,0)$, for $\frac{1}{2} \le x \le \frac{3}{4}$ from $(0,0)$ to $\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta\right)$ and for $\frac{3}{4} \le x \le 1$ from $\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta\right)$ to $(1,0)$.

We can suppose that $((k_1(x), k_2(x))$ is extended linearly from $\left[0, \frac{1}{4}\right]$ to $\left(-\infty, \frac{1}{4}\right]$ and from $\left[\frac{3}{4}, 1\right]$ to $\left[\frac{3}{4}, \infty\right)$. Consider a function

$$\varphi \in C_0^\infty(\mathbb{R}),\ \operatorname{supp}\varphi = [-\delta, \delta], \quad \varphi \ge 0,\ \varphi(x) = \varphi(-x) \forall x,\ \int_{-\infty}^{\infty} \varphi = 1.$$

The existence of a such a function is widely known.

Define

$$\hat{k}_1 = k_1 * \varphi \qquad \hat{k}_2 = k_2 * \varphi.$$

Introduce thes sets

$$A = \{(k_1(x), k_2(x)) : x \in [0,1]\},$$
$$\hat{A} = \{(\hat{k}_1(x), \hat{k}_2(x)) : x \in [0,1]\}.$$

Since the convolution by φ gives an average of the values $k_i(x)$, all points of $\hat{A}$ belongs to the (closed) convex hull of A. Hence $\hat{A}$ lies in the triangle of vertices $(0,0)$, $(1,0)$, $(0,1)$. On the other hand, $\int_{-\infty}^{\infty} x\varphi(x)dx = 0$ implies that $\hat{k}_i(x) = k_i(x)$ whenever k_i varies linearly in $[x-\delta, x+\delta]$. Consequently $\hat{A}$ contains the side $[(1,0)(0,1)]$ of the above mentioned triangle. This means that the function

$$f_1(x,y) = (1-y)\hat{k}_1(x) + y\hat{k}_2(x)$$

is Ky Fan-concave in x; this follows easily from the fact that

$$\forall x_1, x_2 \in [0,1] \quad \forall \lambda \in [0,1] \quad \exists x_3 \in [0,1] :$$
$$\lambda\hat{k}_1(x) + 1(1-\lambda)\hat{k}_1(x_2) \le \hat{k}_1(3),$$
$$\lambda\hat{k}_2(x_1) + (1-\lambda)\hat{k}_2(x_2) \le \hat{k}_2(x_3);$$

see [1], p. 138 or [2] p. 204-205 for more details. Investigate the set

$$C_1 = \{(x_0, y_0) : f_1(x_0, y_0) = \max_{x \in [0,1]} f_1(x, y_0)\}.$$

For given y_0, those values x_0 are involved for which the perpendicular projection of $(\hat{k}_1(x_0), \hat{k}_2)x_0))$ to the line along the vector $(1-y_0, y_0)$ is the farthest from the origin. Keeping in mind what has been proved about the set $\hat{A}$ we see that for $0 \le y_0 < \frac{1}{2}$ only the point $(1,0)$ is projected, for $y_0 = \frac{1}{2}$ the whole segment $[(1,0),(0,-1)]$ and for $\frac{1}{2} < y_0 \le 1$ the only point $(0,1)$. Consequently (using $\operatorname{supp}\varphi = [-\delta, \delta]$)

$$C_1 = \{1\} \times \left[0, \frac{1}{2}\right) \cup \left[0, \frac{1}{4}\right] \times \left\{\frac{1}{2}\right\} \cup \left[\frac{3}{4}, 1\right] \times \left\{\frac{1}{2}\right\} \cup \{0\} \times \left(\frac{1}{2}, 1\right].$$

Considered the function

$$f_2(x,y) = -(x-y)^2;$$

it is obviously concave hence also Ky Fan-concave in y. On the other hand

$$C_2 = \{(x_0, y_0) : f_2(x_0, y_0) = \max_{y \in [0,1]} f_2(x_0, y)\}$$

is the line segment $y = x,\ 0 \le x \le 1$ which does not meet C_1,

$$C_1 \cap C_2 = \emptyset$$

which proves Theorem. □

References

[1] M. Horváth and I. Joó, On Ky Fan convexity, *Matematikai Lapok* **34** (1-3) (1987), 137–140.

[2] I. Joó, Answer to a problem of M. Horváth and A. Sövegjártó, *Annales Univ. Sci. Budapest, Sectio Math.* **29** (1986), 203–207.

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