

A note on primitive classes of arithmetic rings

By L. C. A. van LEEUWEN (Groningen)

Dedicated to the memory of my beloved friend Prof. Andor Kertész

1. In a recent paper [4] MICHLER and WILLE have shown: a primitive class $\mathbf{R}$ of (associative) rings consists of arithmetic rings if and only if $\mathbf{R}$ is generated by a finite set S of finite fields.

A class $\mathbf{R}$ of (associative) rings is primitive if $\mathbf{R}$ is closed with respect to subrings, epimorphic images and direct products, and a ring R is arithmetic if the lattice of its two-sided ideals is distributive. $\mathbf{R}$ is said to be the primitive class generated by S if $\mathbf{R}$ is the smallest primitive class containing S .

It is the purpose of this note to investigate the character of the rings which belong to primitive classes of arithmetic rings.

A class of rings is called a *radical-semisimple class* if it is a radical class for some radical as well as a semisimple class for another radical. Radical-semisimple classes are completely described by STEWART [6]. WIEGANDT showed that a semisimple class is homomorphically closed if and only if it is a radical class. Let $\mathbf{K}_n$ be the class of all associative rings A such that $x^n = x$ for each $x \in A$ ($n = 2, 3, \dots$). Then examples of radical semisimple classes are the classes $\mathbf{K}_n$ for $n = 2, 3, \dots$, [7], [8]. In any class $\mathbf{K}_n$ there is only a finite set F_n of finite fields and $A \in \mathbf{K}_n$ if and only if A is subdirect sum of fields $F \in F_n$, [3].

We show that a primitive class $\mathbf{R}$ of rings consists of arithmetic rings if and only if $\mathbf{R}$ is a radical semisimple class. So examples of primitive classes of arithmetic rings are the classes $\mathbf{K}_n$ for $n = 2, 3, \dots$.

2. A class $\mathbf{R}$ of rings is closed under extensions if $K/A \in \mathbf{R}$, $A \in \mathbf{R} \Rightarrow K \in \mathbf{R}$ whenever A is an ideal in K .

Now we have.

Theorem 1. *Let $\mathbf{R}$ be a primitive class of rings, where $\mathbf{R}$ is not the class of all rings. Then the following are equivalent:*

- (i) $\mathbf{R}$ is closed under extensions
- (ii) $\mathbf{R}$ is a radical class
- (iii) $\mathbf{R}$ is a semisimple class
- (iv) $\mathbf{R}$ is a radical-semisimple class
- (v) $\mathbf{R}$ is a class of arithmetic rings.

PROOF. (i) $\Rightarrow$ (ii). (Theorem 34.1, [9]).

(ii) $\Rightarrow$ (iii). Suppose $\mathbf{R}$ is a radical class. Then $\mathbf{R}$ is closed under complete direct sums and $\mathbf{R}$ is strongly hereditary imply that $\mathbf{R}$ is closed under subdirect sums. Hence $\mathbf{R}$ is a semisimple class ([1]).

(iii) $\Rightarrow$ (iv). A homomorphically closed semisimple class is a radical-semisimple class ([9]).

(iv) $\Rightarrow$ (v). Let $\mathbf{D} = \{A \mid [a] = [a]^2 \text{ for each } a \in A, A \text{ a ring}\}$, where $[a]$ is the subring of A , generated by a . Then if $\mathbf{R} \subseteq \mathbf{D}$, then $\mathbf{R}$ is the class of all rings. ([6]). Hence $\mathbf{R} \subseteq \mathbf{D}$. Now we show that any ring R in $\mathbf{R}$ is an arithmetic ring. Let A, B, C be ideals in R and let $a \in A \cap (B + C)$, $a = b + c$ say, $b \in B, c \in C$. Then $a^2 = ab + ac \in (A \cap B) + (A \cap C)$. Clearly $a^n \in (A \cap B) + (A \cap C)$ for any $n = 2, 3, \dots$. Now $[a] = [a]^2$ implies $a \in [a]^2$ i.e. a can be expressed as a finite sum $a = \sum_{i=2}^r k_i a^i$, $k_i \in \mathbf{Z} \Rightarrow a \in (A \cap B) + (A \cap C)$.

This implies that $A \cap (B + C) = (A \cap B) + (A \cap C)$.

Hence R is an arithmetic ring.

(v) $\Rightarrow$ (i). Suppose $S = K/A \in \mathbf{R}$, $A \in \mathbf{R}$, where A is an ideal in K . Since $\mathbf{R}$ is a primitive class of arithmetic rings, it contains only a finite number of fields ([4], Hilfssatz 3), $\{Z_{p_1}^{a_1}, \dots, Z_{p_n}^{a_n}\} = F$ say.

Any ring in $\mathbf{R}$ is a subdirect sum of finite fields from F ([4], Hilfssatz 4). Conversely, let R be a subdirect sum of finite fields from F . Since $\mathbf{R}$ is closed under subdirect sums (as a primitive class), $R \in \mathbf{R}$.

Hence $\mathbf{R} = \{A \mid A \text{ is a subdirect sum of fields from } F\}$. Then S is a subdirect sum of fields $\{Z_{p_i}^{a_i}\}$, A is a subdirect sum of rings $\{Z_{p_j}^{a_j}\}$, where each $Z_{p_j}^{a_j}$ has an identity.

Then K is a subdirect sum of fields $\{Z_{p_i}^{a_i}\} \cup \{Z_{p_j}^{a_j}\} \subseteq F$ ([5]), hence $K \in \mathbf{R}$.

Thus $\mathbf{R}$ has the extension property.

Note that any class $\mathbf{K}_n$ is a primitive class.

The fields of a given $\mathbf{K}_n$ are exactly those in the corresponding F_n , where $R \in \mathbf{K}_n$ if and only if R is a subdirect sum of fields from F_n . It may be pointed out that for $n \neq m$, $F_n = F_m$ is possible and hence $\mathbf{K}_n = \mathbf{K}_m$, not as asserted in [7] Satz 1, 3. For example,

$$F_4 = \{Z_2, Z_{2^2}\} = F_{10} \quad (\text{cf. [3]}).$$

Since the $\mathbf{K}_n$ are strongly hereditary classes, every F_n is necessarily a strongly hereditary finite set of finite fields. However, not every strongly hereditary finite set of finite fields is some F_n , for $\{Z_2, Z_{2^2}, Z_3\}$ is such a set and is not equal to any F_n and is a proper subset of F_{13} . The precise result is:

a finite field $Z_p^k \in F_n$ if and only if $p^k - 1$ is a divisor of $n - 1$ ([2], Lemma 2).

Remark. JIANG LUH has shown that each ring in $\mathbf{K}_n$ is a direct sum of finitely many p^k -rings (p a prime, k a positive integer). A ring R is called a p^k -ring if there exist a prime p and a positive integer k such that $x^{p^k} = x$ and $px = 0$ for every $x \in R$, [2].

Literature

- [1] E. P. ARMENDARIZ, Closure properties in radical theory, *Pac. J. Math.* **26** (1968), 1—7.
- [2] LUH, JIANG, On the structure of J-rings, *Amer. Math. Monthly* **74** (1967), 164—166.
- [3] L. C. A. VAN LEEUWEN, and T. L. JENKINS, A note on radical semisimple classes, *Publ. Math. (Debrecen)* **21** (1974), 179—184.
- [4] G. MICHLER and R. WILLE, Die primitiven Klassen arithmetischer Ringe, *Math. Z.* **113** (1970), 369—372.
- [5] R. L. SNIDER, Subdirect decompositions of extension rings, *Mich. Math. Journal*, **16** (1969), 225—226.
- [6] P. N. STEWART, Semisimple radical classes, *Pac. J. Math.* **32** (1970), 249—254.
- [7] F. A. SZÁSZ, Beiträge zur Radikaltheorie der Ringe, *Publ. Math. (Debrecen)* **17** (1970), 267—272.
- [8] R. WIEGANDT, Homomorphically closed semisimple classes, *Studia Univ. Babes—Bolyai, Cluj, Ser. Math.-Mech.*, **17**, (2) (1972), 17—20.
- [9] R. WIEGANDT, Radical and semisimple classes of rings, *Queen's papers in pure and applied mathematics*, no. **37**, *Queen's University Kingston, Ontario, Canada* (1974).

(Received December 9, 1974.)