

Prime ideals and zero-divisors in Noetherian-like rings

By K. P. McDOWELL and B. J. MÜLLER (Hamilton, Ont.)

Dedicated to the memory of Professor Andor Kertész

1. Introduction. Let R be a commutative ring with unit. The zero-divisors of a non-zero R -module M will be denoted by $Z(M)$, i.e. $Z(M) = \{r \in R / \text{there exists } 0 \neq m \in M \text{ with } rm = 0\}$. In [4] E. G. EVANS calls R a zero-divisor ring (Z.D. ring) if $Z(R/I)$ is a finite union of prime ideals for each proper ideal I of R . He demonstrates that every non-zero finitely generated module M over a Z.D. ring has the following property:

If I is a finitely generated ideal of R contained in $Z(M)$, then I is the annihilator of some non-zero element of M .

In general, any R -module with this property is called a *pseudo-Noetherian* module. The class of commutative rings determined by the following definition is examined in [7], [8] and [9].

Definition. A *pseudo-Noetherian ring* is a coherent ring which has the property that all of its non-zero finitely presented modules are pseudo-Noetherian.

These rings are interesting primarily because much of the theory of depth and R -sequences developed for local Noetherian rings in [1] and [2] may be extended to local ^{*}) pseudo-Noetherian rings.

It is evident from the above remarks that a coherent Z.D. ring is pseudo-Noetherian. The converse is not necessarily true. For example, a Von Neumann regular ring with infinitely many prime ideals is pseudo-Noetherian but not Z.D. The purpose of this paper is to exhibit a *local* pseudo-Noetherian ring which is not a Z.D. ring.

2. The Example. Let $\mathbf{N}$ represent the set of positive integers. Suppose $\{x_n | n \in \mathbf{N}\}$ is an infinite set of indeterminates and K is a field. Denote by R the subring of $K[[x_n | n \in \mathbf{N}]]$ consisting of all those power series whose expansions contain only finitely many indeterminates.

CLAIM. R is a local pseudo-Noetherian ring which is not a Z.D. ring.

(i) R is a pseudo-Noetherian ring.

^{*}) By "local" ring we mean a possibly non-Noetherian ring with a unique maximal ideal.

For each $n \in \mathbb{N}$ let $R_n = K[[x_1, x_2, \dots, x_n]]$ and notice that R is the union of the chain of rings $(R_n)_{n \in \mathbb{N}}$. If $n' \geq n$, $R_{n'}$ is isomorphic to a direct product of copies of R_n and hence, since R_n is Noetherian, $R_{n'}$ is a flat R_n -module [3, Theorem 2.1]. Furthermore, since the inclusion $R_n \subseteq R_{n'}$ is local, $R_{n'}$ is a faithfully flat R_n -module [6, Section 4. A]. In [8] it is shown that a directed union of pseudo-Noetherian domains, in which all inclusions are faithfully flat, is a pseudo-Noetherian ring.

For every $n \in \mathbb{N}$ each element $r \in R$ has a unique decomposition $r = \varphi_n(r) + x_n \psi_n(r)$ where $\varphi_n(r)$ is that portion of r which is not divisible by x_n and $x_n \psi_n(r)$ is the remainder. The map $\varphi_n: r \rightarrow \varphi_n(r)$ is an idempotent ring endomorphism of R for each $n \in \mathbb{N}$. These decompositions and endomorphisms are useful tools in the following considerations.

To prove that R is not a Z.D. ring, it is necessary to show that there exists an ideal I of R with the property that $Z(R/I)$ is not a finite union of prime ideals. The idea for the proof is derived from a paper of W. HEINZER and J. OHM in which it is demonstrated that a ring R is Noetherian if (and only if) $R[x]$ is a Z.D. ring [5]. Consider the following polynomials defined by iteration.

$$\begin{aligned} f_0 &= x_2 \\ f_n &= x_1 + f_0 f_1 \dots f_{n-1} \quad (n \in \mathbb{N}). \end{aligned}$$

Let I represent the ideal of R generated by $\{x_{n+2} f_1 f_2 \dots f_n \mid n \in \mathbb{N}\}$. It will be shown that $Z(R/I)$ has the desired property.

(ii) $\{f_n \mid n \in \mathbb{N}\} \subseteq Z(R/I)$.

Since $I \subseteq f_1 R$, $x_3 \notin I$ and hence $f_1 \in Z(R/I)$. Now assume $n > 1$ and $x_{n+2} f_1 f_2 \dots f_{n-1}$ is a member of I . Then there exist h_j ($j = 1, 2, \dots, N$) in R such that $x_{n+2} f_1 f_2 \dots f_{n-1} = \sum_{j=1}^N x_{j+2} f_1 f_2 \dots f_j h_j$. Now apply $\varphi_3 \varphi_4 \dots \varphi_{n+1}$ to both sides of this equation and then cancel $f_1 f_2 \dots f_{n-1}$ to obtain $x_{n+2} \in f_n R$. Since this is impossible, $x_{n+2} f_1 f_2 \dots f_{n-1} \notin I$ and therefore $f_n \in Z(R/I)$.

(iii) $x_1 \notin Z(R/I)$

Suppose to the contrary that $x_1 \in Z(R/I)$. Among all $h \in R$ with $h \notin I$ but $x_1 h \in I$, pick one with a representation $x_1 h = \sum_{n=M}^N x_{n+2} f_1 f_2 \dots f_n h_n$ of minimum length $N - M$. Since

$$h - \sum_{n=M}^N x_{n+2} f_1 f_2 \dots f_n \psi_1(h_n) \notin I$$

and

$$x_1 \left(h - \sum_{n=M}^N x_{n+2} f_1 f_2 \dots f_n \psi_1(h_n) \right) = \sum_{n=M}^N x_{n+2} f_1 f_2 \dots f_n \varphi_1(h_n) \in I$$

we may assume in addition that $\varphi_1(h_n) = h_n$ ($M \leq n \leq N$).

Now

$$\begin{aligned} x_1 h &= x_{M+2} f_1 f_2 \dots f_M \left(h_M + \sum_{n=M+1}^N x_{n+2} f_{M+1} f_{M+2} \dots f_n \psi_{M+2}(h_n) \right) + \\ &+ \sum_{n=M+1}^N x_{n+2} f_1 f_2 \dots f_n \varphi_{M+2}(h_n). \end{aligned}$$

For each n , $M+1 \leq n \leq N$,

$$f_{M+1}f_{M+2} \cdots f_n = \varphi_1(f_{M+1}f_{M+2} \cdots f_n) + x_1\psi_1(f_{M+1}f_{M+2} \cdots f_n).$$

Therefore

$$\begin{aligned} x_1 \left(h - x_{M+2}f_1f_2 \cdots f_M \sum_{n=M+1}^N x_{n+2}\psi_1(f_{M+1}f_{M+2} \cdots f_n)\psi_{M+2}(h_n) \right) = \\ = x_{M+2}f_1f_2 \cdots f_M \left[h_M + \sum_{n=M+1}^N x_{n+2}\varphi_1(f_{M+1}f_{M+2} \cdots f_n)\psi_{M+2}(h_n) \right] + \\ + \sum_{n=M+1}^N x_{n+2}f_1f_2 \cdots f_n\varphi_{M+2}(h_n). \end{aligned}$$

From above

$$x_1h = \sum_{n=M}^N x_{n+2}f_1f_2 \cdots f_nh_n \quad \text{with} \quad \varphi_1(h_n) = h_n (M \leq n \leq N).$$

Applying φ_1 to both sides of this equation we obtain $0 = \sum_{n=M}^N x_{n+2}\varphi_1(f_1f_2 \cdots f_n)h_n$ and applying φ_{M+2} on this yields

$$0 = \sum_{n=M+1}^N x_{n+2}\varphi_1(f_1f_2 \cdots f_n)\varphi_{M+2}(h_n).$$

Hence,

$$0 = x_{M+2}\varphi_1(f_1f_2 \cdots f_M)h_M + \sum_{n=M+1}^N x_{n+2}\varphi_1(f_1f_2 \cdots f_n)x_{M+2}\psi_{M+2}(h_n)$$

i.e.,

$$0 = x_{M+2}\varphi_1(f_1f_2 \cdots f_M) \left[h_M + \sum_{n=M+1}^N x_{n+2}\varphi_1(f_{M+1}f_{M+2} \cdots f_n)\psi_{M+2}(h_n) \right].$$

Since $x_{M+2}\varphi_1(f_1f_2 \cdots f_M) \neq 0$, the expression in square brackets is zero. Consequently, for

$$h' = h - x_{M+2}f_1f_2 \cdots f_M \sum_{n=M+1}^N x_{n+2}\psi_1(f_{M+1}f_{M+2} \cdots f_n)\psi_{M+2}(h_n) \quad (h' \notin I)$$

there is a shorter representation

$$x_1h' = \sum_{n=M+1}^N x_{n+2}f_1f_2 \cdots f_n\varphi_{M+2}(h_n); \text{ a contradiction.}$$

(iv) $Z(R/I)$ is not a finite union of prime ideals.

If $Z(R/I)$ were a finite union of prime ideals then two different f_m and f_n ($m > n$) would both lie in one prime ideal. But then $x_1 = f_m - f_0f_1 \cdots f_n \cdots f_{m-1}$ would also lie in that ideal which contradicts (iii).

Hence, R is not a Z.D. ring.

3. Remark. Let $R = K[x_n | n \in \mathbb{N}]_{(x_n \in n | \mathbb{N})}$ be the localization of the polynomial ring in infinitely many indeterminates at the maximal ideal generated by these indeterminates. It is possible to show by a similar argument that R is also a local pseudo-Noetherian ring which is not a Z.D. ring.

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K. P. McDOWELL, Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada.

B. J. MÜLLER, Department of Mathematics, McMaster University, Hamilton, Ontario, Canada.

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