

A Frobenius-type theorem for supersolvable groups

By WANG CAIYUN (Shanxi) and GUO XIUYUN (Shanxi)

Abstract. Frobenius' Theorem for p -nilpotent groups is one of the most fundamental theorems in finite group theory. In this paper a Frobenius-type Theorem for supersolvable groups is given by applying strictly p -closed groups, and some applications are obtained.

Throughout, all groups mentioned are assumed to be finite groups. The terminology and notations employed agree with standard usage.

Let p be a prime. A group G is said to be strictly p -closed whenever G_p , a Sylow p -subgroup of G , is normal in G with G/G_p Abelian of exponent dividing $p - 1$.

Let P be a Sylow p -subgroup of a group G ; Frobenius' Theorem [1, Theorem 10.3.2] states that: a group G is p -nilpotent, if and only if $N_G(P_1)/C_G(P_1)$ is a p -group for every subgroup P_1 of P . If the condition that $N_G(P_1)/C_G(P_1)$ is a p -group is replaced by the weaker condition that $N_G(P_1)/C_G(P_1)$ is a strictly p -closed group, we can obtain a generalization of Frobenius' Theorem for supersolvable groups.

First we prove the following

Theorem 1. *Let G be a p -solvable group, N a normal subgroup of G such that G/N is a p -supersolvable group. If $N_G(P)/C_G(P)$ is strictly p -closed for every p -subgroup P of N , then G is p -supersolvable.*

PROOF. Let K be a minimal normal subgroup of G contained in N . Then K is an elementary Abelian p -group or a p' -group since G is a p -solvable group. Set $\overline{G} = G/K$, and $\overline{N} = N/K$. If K is an elementary Abelian p -group, then, for every p -subgroup $\overline{P} = P/K$ of $\overline{N}$, P is a p -subgroup of N , and so $N_G(P)/C_G(P)$ is strictly p -closed. Since the quotient group of a strictly p -closed group is also a strictly p -closed

group, $(N_G(P)/K)/(C_G(P)K/K)$ is a strictly p -closed group. It follows from $N_G(P)/K = N_{G/K}(P/K)$ and $C_{G/K}(P/K) \geq C_G(P)K/K$ that $N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$ is a strictly p -closed group. If K is a p' -group, then, for every p -subgroup $\overline{P} = H/K$ of $\overline{N}$, $H = PK$, where $P \in \text{Syl}_p H$. By the condition $N_G(P)/C_G(P)$ is strictly p -closed, and so $N_G(P)K/C_G(P)K$ is also strictly p -closed. It is clear that $C_{\overline{G}}(\overline{P}) \geq C_G(P)K/K$. Using [3, Theorem 3.16] $N_{\overline{G}}(\overline{P}) = N_G(P)K/K$ we have that $N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$ is strictly p -closed. Hence we conclude by induction that G/K is p -supersolvable.

If K is a p' -group, then G is p -supersolvable. If K is an elementary Abelian p -group, set $C = C_G(K)$. By the condition G/C is strictly p -closed. Let $A/C \in \text{Syl}_p(G/C)$, then $A/C \triangleleft G/C$, and the semidirect product $A/C \ltimes K$ is a p -group. Hence $Z(A/C \ltimes K) \cap K \neq 1$. Since G/C can act on $Z(A/C \ltimes K) \cap K$, by conjugation and since the action of G/C on K is irreducible we have $Z(A/C \ltimes K) \cap K = K$. Hence the action of A/C on K is trivial and $A/C = 1$. Therefore G/C is Abelian of exponent dividing $p - 1$. By [2, Theorem I.1.4] $|K| = p$, and G is p -supersolvable. The proof of Theorem 1 is complete.

Theorem 2. *Let N be a normal subgroup of a group G , and G/N be a supersolvable group. Then G is a supersolvable group if and only if for every prime $p \mid |N|$, $N_G(P)/C_G(P)$ is a strictly p -closed group for every p -subgroup P of N .*

The proof of Theorem 2 needs the following

Lemma 1. *Let P be a p -subgroup of a group G , and $N_G(P)/C_G(P)$ a strictly p -closed group. If H is a subgroup of G , and $P \leq H$, then $N_H(P)/C_H(P)$ is a strictly p -closed group too.*

PROOF. Since $N_H(P) = H \cap N_G(P)$ and $C_H(P) = H \cap C_G(P)$, we have

$$N_H(P)/C_H(P) = H \cap N_G(P)/H \cap C_G(P) \simeq [H \cap N_G(P)]C_G(P)/C_G(P).$$

Noticing that subgroups of a strictly p -closed group are strictly p -closed groups, $N_H(P)/C_H(P)$ is strictly p -closed.

PROOF of Theorem 2. Assume first that G is a supersolvable group. Let p be a prime, P a p -subgroup of N , $H = N_G(P)$. Since $P \triangleleft H$, we have a chief series of H passing through P :

$$1 = P_0 < P_1 < \cdots < P_s = P \leq \cdots \leq H.$$

As a subgroup of the supersolvable group G , H itself is supersolvable, and so $|P_j/P_{j-1}| = p$ ($j = 1, 2, \dots, s$). By [2, Theorem I.1.4]

$$\text{Aut}_H(P_j/P_{j-1}) \simeq H/C_H(P_j/P_{j-1})$$

is Abelian of exponent dividing $p - 1$. Set $L = \bigcap_{j=1}^s C_H(P_j/P_{j-1})$ and $C = C_G(P)$, then $L \triangleleft H$ and H/L is also Abelian of exponent dividing $p - 1$, and moreover, $L \geq C$. We claim that L/C is a p -group. Suppose to the contrary that some $Cx \in L/C$ has order n relatively prime to p . Let $\alpha \in \text{Aut}(P)$ be the automorphism induced by x , i.e., $\alpha(g) = x^{-1}gx$ ($g \in P$), then the order of α in $\text{Aut}(P)$ divides n , hence it is also relatively prime to p . Also note that $x \in L$ implies $[P_j, \alpha] \leq P_{j-1}$ for $1 \leq j \leq s$, so that [2, Lemma I.1.11] applies to show α is trivial. Hence so is Cx too, proving the claim. It follows that $N_G(P)/C_G(P)$ is strictly p -closed with Sylow p -subgroup L/C .

Suppose now that for every prime $p \mid |N|$, $N_G(P)/C_G(P)$ is a strictly p -closed group for every p -subgroup P of N . Let K be a minimal normal subgroup of G contained in N . Then K is a p -group for some prime p . In fact, assume that p is the smallest prime dividing $|K|$; by Lemma 1 and $(p - 1, |K|) = 1$, $N_K(P)/C_K(P)$ is a p -group for every p -subgroup P of K . Using Frobenius' Theorem [1, Theorem 10.3.2], K has a normal p -complement, say L . Noticing that $L \triangleleft G$, $L < K$ and that K is a minimal normal subgroup of G , we have $L = 1$, and hence K is an elementary Abelian p -group.

Set $\overline{G} = G/K$ and $\overline{N} = N/K$. Similarly to the proof of Theorem 1 we have that for every prime $q \mid |\overline{N}|$, $N_{\overline{G}}(\overline{R})/C_{\overline{G}}(\overline{R})$ is strictly q -closed for every q -subgroup $\overline{Q}$ of $\overline{N}$. Hence we conclude by induction that G/K is supersolvable. By the condition and Theorem 1 G is p -supersolvable. Noticing that K is a minimal normal p -subgroup of G , we have that K is a cyclic group of order p . It follows that G is supersolvable. The proof of Theorem 2 is complete.

Corollary 1. *A group G is supersolvable if and only if, for every prime $p \mid |G|$, $N_G(P)/C_G(P)$ is strictly p -closed for every p -subgroup P of G .*

Theorem 3. *Let N be a normal subgroup of a group G , and G/N a supersolvable group. Then G is supersolvable if and only if, for every prime $p \mid |N|$, $[N_G(P)/C_G(P)]'$ and $[N_G(P)/C_G(P)]^{p-1}$ are p -groups for every p -subgroup P of N .*

From Theorem 2 and the following Lemma 2 Theorem 3 is immediate.

Lemma 2. *A group G is strictly p -closed if and only if G' and G^{p-1} are p -groups.*

PROOF. If G is strictly p -closed, then G/G_p is Abelian, where $G_p \in \text{Syl}_p G$. Hence $G' \leq G_p$ and G' is a p -group. It follows from the exponent of G/G_p dividing $p-1$ that $g^{p-1} \in G_p$ for every $g \in G$, therefore G^{p-1} is also a p -group.

Suppose now that G' and G^{p-1} are p -groups. Let $G_p \in \text{Syl}_p G$. Since $G' \triangleleft G$, we have $G' \leq G_p$ and so $G_p \triangleleft G$ and G/G_p is Abelian. By using that G^{p-1} is a p -group we have $G^{p-1} \leq G_p$. Hence G/G_p is Abelian of exponent dividing $p-1$.

Corollary 2. *A group G is supersolvable if and only if, for every prime $p \mid |G|$, $[N_G(P)/C_G(P)]'$ and $[N_G(P)/C_G(P)]^{p-1}$ are p -groups for every p -subgroup P of G .*

As an application of Theorem 2, we prove the following

Theorem 4. *Let N be a normal subgroup of a group G , and G/N be a supersolvable group. If every minimal subgroup of N is pronormal in G , and either the Sylow 2-subgroups of N are Abelian or every cyclic subgroup of N of order 4 is pronormal in G , then G is supersolvable.*

The proof of Theorem 4 needs the following

Lemma 3. *Let $A_1, A_2, \dots, A_s$; $B_1, B_2, \dots, B_s$ be subgroups of the group G , and $B_i \triangleleft A_i$, ($i = 1, 2, \dots, s$). If A_i/B_i is Abelian of exponent dividing m , then $(A_1 \cap A_2 \cap \dots \cap A_s)/(B_1 \cap B_2 \cap \dots \cap B_s)$ is also Abelian of exponent dividing m .*

PROOF. We only prove Lemma 3 when $s = 2$. Clearly $B_1 \cap B_2 \triangleleft A_1 \cap A_2$. For any $g_1, g_2 \in A_1 \cap A_2$, since A_1/B_1 and A_2/B_2 are Abelian and $g_1(B_1 \cap B_2) = g_1 B_1 \cap g_1 B_2$, we have $g_1 g_2 (B_1 \cap B_2) = g_2 g_1 (B_1 \cap B_2)$, i.e., $A_1 \cap A_2 / B_1 \cap B_2$ is Abelian. From $g_1^m \in B_1, g_1^m \in B_2$ we have $g_1^m \in B_1 \cap B_2$. Hence the exponent of $A_1 \cap A_2 / B_1 \cap B_2$ divides m .

PROOF of Theorem 4. For any prime $p \mid |N|$, if P is a subgroup of N of order p , then $N_G(P)/C_G(P)$ is Abelian of exponent dividing $p-1$ since $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\text{Aut}(P)$. Hence $N_G(P)/C_G(P)$ is strictly p -closed. If P is a cyclic subgroup of N of order 4, it follows from $|\text{Aut}(P)|=2$ that $N_G(P)/C_G(P)$ is Abelian of exponent dividing 2. Hence $N_G(P)/C_G(P)$ is strictly 2-closed.

Let A be any p -subgroup of N , and x be an element of A of order p . Then $\langle x \rangle$ is subnormal in $N_G(A)$. Using [1, exercise 10.3.3] $\langle x \rangle \triangleleft N_G(A)$.

Since $\Omega_1(A) \triangleleft N_G(A) = H$, $C_H(\Omega_1(A)) \triangleleft H$, it is clear that $C = C_G(A) \leq C_H(\Omega_1(A))$. We claim that $C_H(\Omega_1(A))/C$ is a p -subgroup of H/C if $p \neq 2$, or $p = 2$ and A is Abelian. In fact, let $gC \in C_H(\Omega_1(A))/C$ and the order of gC be a p' -number. Noticing that $\langle gC \rangle$ can act on A by conjugation, and that the action of $\langle gC \rangle$ on $\Omega_1(A)$ is trivial, the action of $\langle gC \rangle$ on A is trivial by [3, Theorem 7.26] if $p \neq 2$ or by [4, Theorem 5.2.4] if $p = 2$ and A is Abelian. Hence $gC = C$, i.e., $C_H(\Omega_1(A))/C$ is a p -group. Noticing that $C_H(\Omega_1(A)) = \bigcap_{x \in \Omega_1(A)} (C_H(\langle x \rangle))$, $H \subseteq \bigcap_{x \in \Omega_1(A)} N_H(\langle x \rangle)$ and

that $N_H(\langle x \rangle)/C_H(\langle x \rangle)$ is Abelian of exponent dividing $p - 1$ (when x has order p), $H/C_H(\Omega_1(A))$ is Abelian of exponent dividing $p - 1$ by Lemma 3. Hence $H/C = N_G(A)/C_G(A)$ is strictly p -closed if $p \neq 2$ or $p = 2$ and A is Abelian.

If A is a 2-subgroup of N and A is not Abelian, by considering the subgroup $\Omega_2(A)$ and using [3, Theorem 7.26], similar to the above proof we have that $C_H(\Omega_2(A))/C_G(A)$ is a 2-group, and that $H/C_H(\Omega_2(A))$ is Abelian of exponent dividing 2. Hence $H/C = N_G(A)/C_G(A)$ is a 2-group, and so strictly 2-closed. By Theorem 1 G is supersolvable. The proof of Theorem 4 is complete.

Remark. The statement of Theorem 4 for the case when N has odd order has been proved by M. ASAAD in [5].

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WANG CAIYUN
DEPARTMENT OF MATHEMATICS
SHANXI UNIVERSITY
TAIYUAN, SHANXI 030006
PEOPLE'S REPUBLIC OF CHINA

GUO XIUYUN
DEPARTMENT OF MATHEMATICS
SHANXI UNIVERSITY
TAIYUAN, SHANXI 030006
PEOPLE'S REPUBLIC OF CHINA

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