

On minimal pure subgroups

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In [4], L. FUCHS poses the problem of characterizing the subgroups of an abelian group G , which are intersections of finitely many pure subgroups of G (problem 13 p. 134). In considering this problem, we are naturally brought to think of those subgroups which are contained in no proper pure subgroups of G . We establish, in the first section of this paper, a necessary and sufficient condition for a pure subgroup K of G to be a minimal pure subgroup containing a subgroup N . In the second section, we make some applications of this characterization to obtain new and shorter proofs of some results contained in [6] and in [8], as well as some new results connecting these ideas with direct sums of cyclic groups. The notation is that of [4] all groups are abelian, Z^+ denotes the set of non-negative integers, and p denotes a fixed positive prime number. The symbol $\oplus_c$ denotes direct sums of cyclic groups.

1. Minimal pure subgroups

Definition 1.1. A subgroup N of a group G is said to be almost-dense in G (abbreviated a.d.) if, for every pure subgroup K of G containing N , G/K is divisible.

Notation 1.2. Let K be any primary group, for every $n \in Z^+$ we let

$$K_n = (p^n K)[p] = \{x \in p^n K \mid px = 0\}.$$

Theorem 1.3. *Let N be a subgroup of a primary group G . There is no proper pure subgroup of G containing N if and only if N is almost-dense in G , and $N \supset G_n$ for some $n \in Z^+$.*

PROOF. Suppose N is a.d. and $N \supset G_n$ for some $n \in Z^+$. Then, every pure subgroup K of G , containing N , contains also $p^n G$ (see [1], Corollary 3, p. 342). Thus, G/K is at the same time divisible and bounded. And therefore $G/K=0$, i.e. $G=K$. Conversely, if no proper pure subgroup of G contains N , clearly N is a.d. in G . Also, by Lemma 4.1 and theorem 3.7 in [2], $N \supset G_n$ for some $n \in Z^+$.

In view of the preceding theorem, we need only characterize almost dense subgroups of G . We need the following two lemmas.

Lemma 1.4. *Let G be a primary group and S a subgroup of $G[p]$ such that $S \cap p^n G = 0$, for some $n \in \mathbb{Z}^+$. Then there exists a pure subgroup K of G such that $K[p] = S$. Furthermore*

$$(K \oplus p^n G)/p^n G \text{ is a pure subgroup of } G/p^n G.$$

PROOF. S can be extended to a $p^n G$ -high subgroup H of G . H is then, pure and bounded (see [4] Theorem 27.7) and there exists a pure subgroup K of H such that $K[p] = S$ (see [7] ex. 18, a). It is easy to check that $K \oplus p^n G/p^n G$ is pure in $G/p^n G$.

Lemma 1.5. *Let N be a subgroup of a primary group G , such that for some $n \geq 1$, $N + p^n G \cong G_{n-1}$. Then, there exists a proper pure subgroup R of G such that:*

$$R \supseteq N + p^n G, \quad \text{and,} \quad N + p^n R \supseteq R_{n-1}.$$

PROOF. Let $G_{n-1} = T \oplus S$, where $T = (N + p^n G) \cap G_{n-1}$. Since $S \cap p^n G = 0$ there exists, by lemma 1.4, a pure subgroup K of G , such that : $K[p] = S$, and $(K \oplus p^n G)/p^n G$ is a pure subgroup of $G/p^n G$, whose socle is $(S \oplus p^n G)/p^n G$. Now, we have:

$$0 = p^n(G/p^n G) \subseteq (S \oplus p^n G)/p^n G \subseteq p^{n-1}(G/p^n G).$$

Thus, $(K \oplus p^n G)/p^n G$ is an absolute summand of $G/p^n G$ (see [1] p. 344). Let $R/p^n G$ be a complementary summand of $(K \oplus p^n G)/p^n G$, in $G/p^n G$, containing $(N + p^n G)/p^n G$. Then $G = R \oplus K$, where $K \neq 0$. Now, since $R \supseteq p^n G$, we have $p^n G = R \cap p^n G = p^n R$. Also $G_{n-1} \supseteq R_{n-1} = R \cap G_{n-1} \supseteq T$, therefore $R_{n-1} = T \oplus (S \cap R_{n-1})$, but $S \cap R_{n-1} = 0$ so $R_{n-1} = T$, and we have

$$N + p^n R = N + p^n G \supseteq T = R_{n-1}, \text{ as stated.}$$

Now we use these lemmas to obtain a characterization of almost dense subgroups of a primary group G .

Theorem 1.6. *A subgroup N of a primary group G is almost-dense in G if and only if*

$$(*) \quad N + p^n G \supseteq G_{n-1} \text{ for all } n \geq 1.$$

PROOF. Suppose N satisfies $(*)$ and K is a pure subgroup of G containing N . If G/K is not divisible it must have a cyclic summand $R/K \neq 0$ (see [7] theorem 9). Now, $G/K = H/K \oplus R/K$ and $G/H \simeq R/K$ is finite so that for some $n \in \mathbb{Z}^+$, $p^n G \subseteq H$. Thus, $H \supseteq N + p^n G \supseteq G_{n-1}$ and H being pure we have $H \subseteq p^{n-1}G$. After a finite number of steps we see that $H \supseteq N + pG \supseteq G[p]$, i.e. $H = G$, and $R/K = 0$. This is a contradiction, therefore G/K is divisible and N is a.d. in G . Conversely, if $(*)$ is not satisfied then we are in the situation of lemma 1.5 and N cannot be a. d. since there exists a proper pure subgroup $R \supseteq N + p^n G$ and $G/R \neq 0$ is bounded. Therefore if N is a.d. $(*)$ is satisfied.

Combining theorem 1.3 and theorem 1.6 we obtain.

Theorem 1.7. *Let K be a pure subgroup of a primary group G , containing a subgroup N of G . K is a minimal pure subgroup of G containing N if and only if*

$$N \supseteq K_n \text{ for some } n \in \mathbb{Z}^+ \text{ and } N + r^n K \supseteq K_{r-1} \text{ for } 1 \leq r.$$

2. Some applications

We apply now the results in the first section to obtain the following useful criterion.

Theorem 2.1. *Let N be a subgroup of a primary group G . N is contained in a minimal pure subgroup of G if and only if there exists a pure subgroup K of G such that*

$$K \supseteq N \supseteq K_n \text{ for some } n \in \mathbb{Z}^+.$$

PROOF. If N is contained in a minimal pure subgroup of G , the result follows from theorem 1.7. Conversely, suppose that there exists a pure subgroup K of G such that $K \supseteq N \supseteq K_n$ for some $n \in \mathbb{Z}^+$. If $n=0$, K itself is minimal pure containing N . If $n \geq 1$, for every pure subgroup R of K containing N , define.

$$E(R) = \{r \geq 1 \mid N + p^r R \not\supseteq R_{r-1}\}$$

and set $m(R) = 0$ if $E(R) = \emptyset$

$$m(R) = \max \{x \in E(R)\} \text{ if } E(R) \neq \emptyset$$

note that, $m(R) \leq n$ and therefore there exists a pure subgroup H of G containing N for which $m(H)$ is minimal. By lemma 1.5 we see that $m(H)=0$, i.e.

$$H \supseteq N \supseteq H_n \text{ and } N + p^r H \supseteq H_{r-1} \text{ for all } r \geq 1.$$

Thus by theorem 1.7, H is a minimal pure subgroup containing N . This completes the proof.

Remark. The preceding criterion was first, to our knowledge, formulated by J. D. MOORE in his doctoral dissertation ([8] p. 45). His proof however, was based on an unproved conjecture ([8] p. 41) whose validity has not been established anywhere that we know of.

Corollary 2.2. ([6]) In a bounded primary group every subgroup is contained in a minimal pure subgroup.

To conclude this section, we look at the important special cases where $G/N = \oplus_c$, or where N is contained in a basic subgroup of G . In [3], we proved that if a subocle S of a primary group G supports a pure subgroup K of G , then $G/S = \oplus_c$ implies $G/K = \oplus_c$. Our next result generalizes this fact.

Theorem 2.3. *Let N be a subgroup of a primary group G , such that $G/N = \oplus_c$. If K is a minimal pure subgroup of G containing N then $G/K = \oplus_c$.*

PROOF. We know that $N \supseteq K_n$ for some $n \in \mathbb{Z}^+$. Now, since K is pure in G , $p^n(G/K)[p] = (G_n + K)/K$, and the natural homomorphism from G/N onto G/K maps $(G_n + N)/N$ onto $(G_n + K)/K$. This map is height preserving, indeed for $g \in G_n$

$$h(g + K) \leq n \text{ and if } g + K = p^m x + K \text{ where } m \leq n,$$

the purity of K allows us to choose $p^m x \in G[p]$, thus $g - p^m x \in K[p]$. Now, $h_G(g - p^m x) \leq n$ and $h_K(g - p^m x) \leq n$, therefore $g - p^m x \in K_n \subset N$, so that $g + N = p^m x + N$ and $h(g + K) \leq h(g + N)$. Since $(G_n + N)/N$ is the union of an ascending chain of subgroups of bounded height in G/N , we conclude that $(G_n + K)/K$ has the same property

in G/K . Thus, $p^n(G/K) = \oplus_c$ (see [7] theorem 12) and $G/K = \oplus_c$ (see [4] prop. 18, 3 p. 92). This completes the proof.

A close examination of the preceding proof yields the following.

Corollary 2.4. Let N be a subgroup of a primary group G , such that $G/N = \oplus_c$ and let K be a pure subgroup of G , such that: $K \supseteq N \supseteq K_n$, for some $n \in \mathbb{Z}^+$, then K is a summand of G .

The final result in this section is in some sense dual to theorem 2.4.

Theorem 2.5. Let N be a subgroup of a basic subgroup B of a primary group G . If N is contained in a minimal pure subgroup K of G then $K = \oplus_c$.

PROOF. Since $N \subseteq B$, N satisfies the conditions of theorem 29.5 p. 99 in [5], and since K is pure, N can be expanded to a basic subgroup H of K . Since K is minimal pure containing N , $H = K$ and therefore $K = \oplus_c$ as stated.

References

- [1] BENABDALLAH (KHALID) and IRWIN (JOHN M.). On N -high subgroup of abelian groups. *Bull. Soc. Math. France* **96** (1968), 337—346.
- [2] ——— On Quasi-essential subgroups of primary abelian groups, *Can. J. Math.* **12** (1970), 1176—1184.
- [3] ——— Pure N -high subgroups, p -adic topology and direct sums of cyclic groups. *Can. J. Math.* **26** (1974), 322—327.
- [4] FUCHS (LÁSZLÓ) Infinite abelian groups. Vol 1. New York-London (1970).
- [5] ——— Abelian groups. Pergamon press (1967).
- [6] HILL (PAUL) and MEGIBBEN (CHARLES), Minimal pure subgroups in primary groups, *Bull. Soc. Math. France* **92** (1964), 251—257.
- [7] KAPLANSKI (IRVING), Infinite abelian groups, *Univ. of Mich. Press, Ann. Arbor* (1954).
- [8] J. D. MOORE, On purity in abelian groups, *Thesis, Syracuse University* (1969).

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