

Extensions of almost diffuse, countably almost diffuse and c_0 -operators

By E. TARAFDAR (St. Lucia, Qld)

In [1] FOIAIS and SINGER have introduced the classes of almost diffuse and countably almost diffuse operators mapping a space $C([0, 1])$ into a Banach space X .

WHITLEY [2] has shown the validity of the results of [1] when S is an arbitrary compact Hausdorff space and has also introduced a class of operators called c_0 -operators mapping a space $C(S)$ into a Banach space. All these classes of operators include compact and weakly compact operators and have many other properties (see [1] and [2]). The following wellknown result is due to LINDENSTRAUSS [3]:

The second conjugate X^{**} of a Banach space is a P_λ space $\Leftrightarrow$ each weakly compact (resp. compact) operator from X into a Banach space Y has a weakly compact (resp. compact) extension $\tilde{T}$ from $Z(Z \supset X)$ into Y with $\|\tilde{T}\| \leq \lambda \|T\|$. (For definition of a P_λ space, $\lambda \geq 1$ see DAY [4], p. 94.)

Since almost diffuse, countably almost diffuse and c_0 -operators are defined only on the spaces $C(S)$ and it is wellknown that $C(S)^{**}$ is a P_1 space for any compact Hausdorff space S (e.g. see GOODNER [5], p. 106), an analogue of the above result of Lindenstrauss for these operators will be the following: Each almost diffuse (resp. countably almost diffuse, resp. c_0 -) operator T mapping a space $C(S)$ into a Banach space X has an almost diffuse (resp. a countably almost diffuse, resp. a c_0 -) extension $\tilde{T}$ mapping $C(Q)$ ($C(Q) \supset C(S)$) into X with $\|\tilde{T}\| = \|T\|$. In this note we prove this result in the particular case when $Q = A$, the unit ball of $C(S)^*$.

We now state the the following definitions as given in [1] and [2].

S is a compact Hausdorff space and X any Banach space. The oscillation of a bounded linear operator $T: C(S) \rightarrow X$ at a point $s \in S$ is denoted by $W_T(s) = W(T, s)$ and is defined to be the supremum over all positive α satisfying the following: for every neighborhood U of s there is a function f , of norm ≤ 1 in $C(S)$, which vanishes outside of U with $\|Tf\| \geq \alpha$.

Let $T: C(S) \rightarrow X$ be a bounded linear operator. Then T is almost diffuse if the set of diffusion points of T , $D(T) = W_T^{-1}(0)$, is dense. T is countably almost diffuse if the set of concentration points, $\gamma(T) = \{s: s \in S, W_T(s) > 0\}$, is countable and T is c_0 -operator if for each $\varepsilon > 0$, the set $\{s: s \in S, W_T(s) > \varepsilon\}$ is finite.

Let Y be a Banach space and A be the unit ball in Y^* . Then A is a compact Hausdorff space in the weak*-topology (the Y -topology of Y^* in the terminology of DUNFORD and SCHWARZ [6]). Y is isometrically isomorphic to a closed subspace of the Banach space $C(A)$ where π is the corresponding isometric isomorphism

of Y onto a closed subspace of $C(A)$ and is defined as follows: for each $y \in Y$, $\varkappa(y) =$ the restriction of $\varkappa_1(y)$ to A where $\varkappa_1$ is the natural embedding of Y into Y^{**} (see DUNFORD and SCHWARTZ [6], con. 3, p. 424). We first prove the following lemma:

Lemma. *For every bound linear operator $T: C(S) \rightarrow X$ mapping a space $C(S)$ into an arbitrary Banach space X , there exists a bounded linear operator $\tilde{T}: C(A) \rightarrow X$ such that $\tilde{T}\varkappa = T$ and $\|\tilde{T}\| = \|T\|$ where A is the unit ball in $C(S)^*$ and $\varkappa$ has the corresponding meaning as explained above. In other words every bounded linear operator $T: C(S) \rightarrow X$ has a bounded linear extension $\tilde{T}: C(A) \rightarrow X$ with $\|\tilde{T}\| = \|T\|$.*

PROOF. By lemma 7 of DUNFORD and SCHWARTZ ([6], p. 442) there is a homeomorphism λ of S onto a subset of A where A has weak*-topology and λ is defined by $\lambda: s \rightarrow x_s^*$ for each $s \in S$ and x_s^* is defined by $x_s^*(f) = f(s)$, $f \in C(S)$. We now construct our required operator $\tilde{T}$.

Let f_1 be an arbitrary element of $C(A)$. Then the restriction $\tilde{f}_1$ of f_1 to $\lambda(S)$ is a continuous map on $\lambda(S)$ where $\lambda(S)$ has the relative weak*-topology. Thus the composition $\tilde{f}_1\lambda$ is a continuous map on S . We set $\tilde{T}f_1 = T\tilde{f}_1\lambda$ for each $f_1 \in C(A)$. Clearly $\tilde{T}$ is well defined and linear. We next prove the boundedness of $\tilde{T}$. We observe that $\|\tilde{f}_1\lambda\| = \sup_{s \in S} |\tilde{f}_1\lambda(s)| = \sup_{x^* \in \lambda(S)} |\tilde{f}_1(x^*)| \leq \sup_{x^* \in A} |f_1(x^*)| = \|f_1\|$. Hence $\|\tilde{T}f_1\| = \|T\tilde{f}_1\lambda\| \leq \|T\| \|\tilde{f}_1\lambda\| \leq \|T\| \|f_1\|$. Hence $\tilde{T}$ is bounded and moreover

$$(1) \quad \|\tilde{T}\| \leq \|T\|.$$

Next, we prove that $\tilde{T}\varkappa = T$. Let $f \in C(S)$ be arbitrary and $\varkappa f = g$. We first see that $\tilde{g}\lambda = f$ where $\tilde{g}$ is the restriction of g to $\lambda(S)$. For each $s \in S$,

$$\begin{aligned} \tilde{g}\lambda(s) &= \tilde{g}(x_s^*) = g(x_s^*) \text{ as } x_s^* \in \lambda(S), \\ &= \varkappa f(x_s^*) = x_s^*(f) \text{ by definition of } \varkappa \\ &= f(s) \text{ by definition of } x_s^*. \end{aligned}$$

Thus $\tilde{g}\lambda = f$. Hence $\tilde{T}\varkappa f = \tilde{T}g = T(\tilde{g}\lambda) = Tf$. Now $\|T\| = \|\tilde{T}\varkappa\| \leq \|\tilde{T}\| \|\varkappa\| = \|\tilde{T}\|$. Combining this with (1), we obtain $\|\tilde{T}\| = \|T\|$. This completes the proof.

Let $T: C(S) \rightarrow X$ be any bounded linear operator. We suppose that $\tilde{T}$ has been obtained as in the above lemma. Remembering all the notations used in the proof of the above lemma we note the following facts:

(1) Every point $y \in [\lambda(S)]^c$ (complement of $\lambda(S)$ in A) is a diffusion point of $\tilde{T}$, i.e., $W_{\tilde{T}}(y) = 0$ for each $y \in [\lambda(S)]^c$.

PROOF. Let $\varepsilon > 0$ be any number. Now $0 = [\lambda(S)]^c$ is a neighborhood of y in A . Let f be any function in $C(A)$ such that $\|f\| \leq 1$ and f vanishes outside of 0 . Obviously $\tilde{f}\lambda(s) = 0$ for each $s \in S$, where $\tilde{f}$ is the restriction of f to $\lambda(S)$ as before. Hence $\|\tilde{T}f\| = \|T\tilde{f}\lambda\| = 0 < \varepsilon$. Thus $(W_{\tilde{T}})_y = 0$. (Notice that same 0 works for each $\varepsilon > 0$).

(2) If y is a concentration point of $\tilde{T}$, i.e., $y \in \gamma(\tilde{T})$, then (i) $y \in \gamma(\tilde{T}) \cap \lambda(S)$, (ii) $s = \lambda^{-1}(y) \in \gamma(T)$ and (iii) furthermore if $W_{\tilde{T}}(y) > \alpha$, $\alpha > 0$, then $W_T(s) > \alpha$.

PROOF. (i) follows from (1). Let $W_{\tilde{T}}(y) > \varepsilon$, $\varepsilon > 0$. Let N be a neighborhood of s in S . Then $\lambda(N)$ is a neighborhood of y in $\lambda(S)$. We can find a neighborhood Q of y in A such that $Q \cap \lambda(S) = \lambda(N)$. Since $W_{\tilde{T}}(y) > \varepsilon$. There exists a function f in $C(A)$ such that f vanishes outside of Q , $\|f\| \leq 1$ and $\|\tilde{T}f\| > \varepsilon$. Let $\tilde{f}$ be the restriction

of f to $\lambda(S)$. Obviously $\tilde{f}$ vanishes outside of $\lambda(N)$ in $\lambda(S)$ (i.e. outside of $\lambda(N)$ relative to $\lambda(S)$). We consider the function $\tilde{f}\lambda$ in $C(S)$. Clearly $\tilde{f}\lambda$ vanishes outside of N and moreover as in the proof of the lemma, $\|\tilde{f}\lambda\| \leq \|f\| \leq 1$. Also as in the proof the lemma $\|T\tilde{f}\lambda\| = \|Tf\| > \varepsilon$. Thus it follows that $W_T(s) > \varepsilon$ and $s \in \gamma(T)$. Hence both (ii) and (iii) are proved.

(3) If s is a diffusion point of T , i.e., if $s \in D(T)$ then $\lambda(s) \in D(\tilde{T})$.

PROOF. That $\lambda(s) \in D(\tilde{T})$ follows directly from (2). We are now in a position to prove the following theorem:

Theorem. For each almost diffuse (resp. countably almost diffuse, resp. c_0 -) operator $T: C(S) \rightarrow X$ mapping a space $C(S)$ into an arbitrary Banach space X , there is an almost diffuse (resp. a countably almost diffuse, resp. a c_0 -) operator $\tilde{T}: C(\Lambda) \rightarrow X$ such that $\tilde{T}\lambda = T$ and $\|\tilde{T}\| = \|T\|$, where Λ and λ have the same meaning as in the above lemma.

PROOF. By the above lemma, for each bounded linear operator $T: C(S) \rightarrow X$, there is a bounded linear operator $\tilde{T}: C(\Lambda) \rightarrow X$ such that $\tilde{T}\lambda = T$ and $\|\tilde{T}\| = \|T\|$. We prove that $\tilde{T}$ is the required operator in each case.

Let T be almost diffuse. Then $D(T)$ is dense in S . Hence $\lambda(D(T))$ is dense in $\lambda(S)$ as λ is a homeomorphism. Evidently $\lambda(D(T)) \cup [\lambda(S)]^c$ is dense in Λ . By using the facts (1) and (3), we have $\lambda D(T) \cup [\lambda(S)]^c \subseteq D(\tilde{T})$. Hence $D(\tilde{T})$ is dense in Λ . Thus $\tilde{T}$ is almost diffuse.

Next, let T be countably almost diffuse. Then $\gamma(T)$ is countable. By the fact 2 (ii) it follows that $\lambda^{-1}(\gamma(\tilde{T})) \subseteq \gamma(T)$. Now since λ is a homeomorphism, it follows that $\gamma(\tilde{T})$ is countable. Hence $\tilde{T}$ is countably almost diffuse.

Finally, let T be c_0 -operator. If possible, suppose that $\tilde{T}$ is not a c_0 -operator. Then there is a number $\alpha > 0$ such that the set $A = \{y: y \in \Lambda, W_{\tilde{T}}(y) > \alpha\}$ is infinite. By the fact 2 (i), $A = \{y: y \in \lambda(S), W_{\tilde{T}}(y) > \alpha\}$. Now $\lambda^{-1}(A) = \{s: s \in S, \lambda(s) \in A\} = \{s: s \in S, \lambda(s) \in A, W_T(s) > \alpha\}$, by 2 (iii), $\subseteq \{s: s \in S, W_T(s) > \alpha\}$. Since $\lambda^{-1}(A)$ is infinite, λ being a homeomorphism, it follows that $\{s: s \in S, W_T(s) > \alpha\}$ is infinite. This contradicts that T is a c_0 -operator. Hence $\tilde{T}$ is a c_0 -operator.

Remark. Note that all the above results will also hold if Λ is replaced by the weak*-closure of the set of extremal points of Λ .

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UNIVERSITY OF QUEENSLAND,
ST. LUCIA, QLD.

(Received 12. February, 1974.)