

Some remarks on a generalization of the Stieltjes transform

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1. The integral equation

$$(1) \quad \varphi\{f(x) : p\} = \int_0^{\infty} (x+p)^{-1} f(x) dx$$

is said to define the Stieltjes transform of a function $f(x)$, provided that the integral on the right-hand side exists. We say that $\varphi\{f(x):p\}$ is the image of $f(x)$ under the kernel $(x+p)^{-1}$. A natural generalization of (1) is given in the literature by means of the integral equation (cf. [8], p. 30)

$$(2) \quad \varphi^{\lambda}\{f(x) : p\} = \int_0^{\infty} (x+p)^{-\lambda} f(x) dx, \quad \lambda > 0,$$

and whenever this integral exists, $\varphi^{\lambda}\{f(x):p\}$ is called the generalized Stieltjes transform of $f(x)$ of order λ .

Recently, J. L. ARORA [1] has introduced a further generalization of (2) given by

$$(3) \quad \varphi_m^{\lambda}\{f(x) : p\} = \int_0^{\infty} (x^m + p^m)^{-\lambda} f(x) dx, \quad \lambda, m, p > 0,$$

provided that the second member of (3) exists. Evidently, one has

$$(4) \quad \varphi_1^{\lambda}\{f(x) : p\} = \varphi^{\lambda}\{f(x) : p\} \quad \text{and} \quad \varphi_1^1\{f(x) : p\} = \varphi\{f(x) : p\}, \quad \lambda, p > 0,$$

in the notations of (1), (2) and (3).

It may be of interest to remark that the generalization (3) of the integral transform (2) is rather trivial; indeed it is easily verified that

$$(5) \quad \varphi_m^{\lambda}\{f(x) : p\} = m^{-1} p^{1-m\lambda} \varphi^{\lambda}\{x^{(1-m)/m} f(px^{1/m}) : 1\},$$

where $\lambda, m, p > 0$.

The linear relationship (5) between the generalized Stieltjes transform (2) and its subsequent variation (3) studied by Arora [1] can be used to convert readily available tables of $\varphi^{\lambda}\{f(x):p\}$ into those of $\varphi_m^{\lambda}\{f(x):p\}$. As an illustration, we notice that if

$$(6) \quad f(x) = H_{u,v}^{l,n} \left[z x \left| \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_u, A_u) \\ (\beta_1, B_1), \dots, (\beta_v, B_v) \end{matrix} \right. \right],$$

where $H_{u,v}^{l,n}[z]$ denotes the familiar H -function (cf., e.g., [5], p. 214, Eq. (2.4)), $A_j > 0, j=1, \dots, u, B_j > 0, j=1, \dots, v$, then (cf. [2], p. 237)

$$(7) \quad \varphi^\lambda \{f(x) : p\} = \frac{p^{1-\lambda}}{\Gamma(\lambda)} H_{u+1,v+1}^{l+1,n+1} \left[zp \left| \begin{matrix} (0, 1), (\alpha_1, A_1), \dots, (\alpha_u, A_u) \\ (\lambda-1, 1), (\beta_1, B_1), \dots, (\beta_v, B_v) \end{matrix} \right. \right],$$

provided

$$(8) \quad \begin{cases} \lambda > 1 + \max_{1 \leq j \leq n} \{\operatorname{Re}[(\alpha_j - 1)/A_j]\}, & \min_{1 \leq j \leq l} \{\operatorname{Re}(\beta_j/B_j)\} > -1, \\ |\arg(z)| < \frac{1}{2} \pi \Delta, & \Delta \equiv \sum_{j=1}^n A_j - \sum_{j=n+1}^u A_j + \sum_{j=1}^l B_j - \sum_{j=l+1}^v B_j > 0. \end{cases}$$

Making use of (5) and (7) we have

$$(9) \quad \varphi_m^\lambda \{f(x) : p\} = \frac{p^{1-m\lambda}}{m\Gamma(\lambda)} H_{u+1,v+1}^{l+1,n+1} \left[zp \left| \begin{matrix} \left(1 - \frac{1}{m}, \frac{1}{m}\right), (\alpha_1, A_1), \dots, (\alpha_u, A_u) \\ \left(\lambda - \frac{1}{m}, \frac{1}{m}\right), (\beta_1, B_1), \dots, (\beta_v, B_v) \end{matrix} \right. \right],$$

valid under conditions in (8) when $f(x)$ is given by (6).

We remark in passing that a large variety of special functions which occur in problems of applied mathematics and mathematical analysis can be expressed in terms of the H -function which reduces, when $A_j = B_k = 1, j=1, \dots, u; k=1, \dots, v$, to the relatively more familiar G -function and which includes, as a special case, the generalized hypergeometric ${}_u\Psi_v[z]$ function of Wright. Thus the transform pair given by (6) and (9) would apply not only to the simpler special functions of mathematical physics such as the Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, the Bessel functions $J_\nu(z)$ and $I_\nu(z)$, the Whittaker function $M_{\mu,\nu}(z)$, the Wright function $J_\nu^\mu(z)$, the classical orthogonal polynomials of Jacobi, Hermite, Laguerre, etc., which are particular cases of the generalized hypergeometric function ${}_uF_v[z]$ or ${}_u\Psi_v[z]$, but also to the more involved Bessel functions $K_\nu(z)$ and $Y_\nu(z)$, the Whittaker function $W_{\mu,\nu}(z)$, their various combinations and other related functions. Indeed, by making use of extensive tables of functions (or their various combinations) expressible in terms of the G -function or the generalized hypergeometric functions (see, for instance, [2], pp. 434—444; see also [4], pp. 209—234), one can deduce from (6) and (9) a fairly large table of these functions and their respective images under the „generalized” Stieltjes transform (3).

2. Since

$$(10) \quad (x+p)^{-\lambda} = p^{-\lambda} {}_1F_0[\lambda; -; -x/p],$$

it would seem natural to consider further generalizations of the generalized Stieltjes transform (2) by replacing the hypergeometric ${}_1F_0$ function in the kernel by the Gauss function

$$(11) \quad {}_2F_1[\lambda, \mu; \nu; -x/p],$$

or more generally, by the hypergeometric function

$$(12) \quad {}_uF_v[\alpha_1, \dots, \alpha_u; \beta_1, \dots, \beta_v; -x/p],$$

or indeed by the H -function

$$(13) \quad H_{u,v}^{l,n} \left[\frac{x}{p} \middle| \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_u, A_u) \\ (\beta_1, B_1), \dots, (\beta_v, B_v) \end{matrix} \right].$$

Admittedly, the literature is full of generalizations in these directions, and we content ourselves by referring the interested readers to the works of R. S. VARMA [7], J. M. C. JOSHI [3], and R. SWAROOP [6], where kernels of type (11) have been used to generalize (2). As a special case of the results of these papers one can easily deduce a complex inversion formula for the generalized Stieltjes transform (2); thus, under certain conditions, we have (cf. [8], p. 31, Theorem 8.1)

$$(14) \quad f(x) = \frac{\Gamma(\lambda)}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \frac{\Phi^\lambda(s) x^{\lambda-s-1}}{\Gamma(s)\Gamma(\lambda-s)} ds, \quad \tau > 0,$$

where, for convenience, $\Phi^\lambda(s)$ denotes the Mellin transform of $\varphi^\lambda\{f(x):p\}$ given by (2).

More specifically, we have

$$(15) \quad \Phi^\lambda(s) = \int_0^\infty p^{s-1} \varphi^\lambda\{f(x):p\} dp,$$

and on substituting from (2) it is seen fairly simply that

$$(16) \quad \Phi^\lambda(s) = \frac{\Gamma(s)\Gamma(\lambda-s)}{\Gamma(\lambda)} \int_0^\infty x^{s-\lambda} f(x) dx,$$

provided this last integral is convergent. Similarly, if $\Phi_m^\lambda(s)$ denotes the Mellin transform of $\varphi_m^\lambda\{f(x):p\}$, we have

$$(17) \quad \Phi_m^\lambda(s) = \frac{\Gamma(s/m)\Gamma(\lambda-s/m)}{m\Gamma(\lambda)} \int_0^\infty x^{s-m\lambda} f(x) dx.$$

Now an elementary change of the variable of integration in (16) would yield

$$(18) \quad \Phi^\lambda(s/m) = \frac{m\Gamma(s/m)\Gamma(\lambda-s/m)}{\Gamma(\lambda)} \int_0^\infty x^{s-m\lambda} \{x^{m-1}f(x^m)\} dx,$$

which, in conjunction with (17), exhibits the obviously well-anticipated fact that the inversion formula (14) can be rewritten at once in terms of the "generalized" Stieltjes transform (3) if we replace s by s/m , $x^{m-1}f(x^m)$ by $f(x)$, and $\Phi^\lambda(s/m)$ by $m^2\Phi_m^\lambda(s)$. Thus, as a trivial variation of (14), we obtain a complex inversion formula for (3) given by

$$(19) \quad f(x) = \frac{m\Gamma(\lambda)}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \frac{\Phi_m^\lambda(s) x^{m\lambda-s-1}}{\Gamma(s/m)\Gamma(\lambda-s/m)} ds, \quad \tau > 0,$$

where, as also in (14), $f(x)$ must be replaced by $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at a point $t=x$ such that

$$(20) \quad \lim_{t \uparrow x} f(t) = f(x+0) \neq f(x-0) = \lim_{t \downarrow x} f(t).$$

This formula (19) is substantially what is contained in Theorem 2, p. 108 of ARORA [1] who does also state (14) as a „special” case of (19) when $m=1$. One can similarly verify that almost every property of the “generalized” Stieltjes transform (3), given by Arora [op. cit.], would follow rather trivially from the corresponding property of the generalized Stieltjes transform (3) or its nontrivial generalizations studied systematically in the literature (cf., e.g., [3], [6] and [7]).

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