

On a problem of Graham

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GRAHAM stated the following conjecture: Let p be a prime and $a_1, \dots, a_p$ p non-zero residues (mod p). Assume that if $\sum_{i=1}^p \varepsilon_i a_i$, $\varepsilon_i = 0$ or 1 (not all $\varepsilon_i = 0$) is a multiple of p then $\sum_{i=1}^p \varepsilon_i$ is uniquely determined. The conjecture states that in this case there are only two distinct residues among the a 's.

We are going to prove this conjecture for all sufficiently large p , in fact we will prove a sharper result. To extend our proof for the small values of p would require considerable computation, but no theoretical difficulty.

Our proof is surprisingly complicated and we are not convinced that a simpler proof is not possible, but we could not find one.

First we prove

Theorem 1. *Let η_0 be sufficiently small, $\eta < \eta_0$, $p > p_0(\eta)$: $A = \{a_1, \dots, a_l\}$, $l > \eta^{1/10} p$ is a set of non-zero residues mod p . Assume that for every t the number of indices i satisfying $a_i \equiv t \pmod{p}$ is less than $\eta \cdot p$. Then*

$$\sum_{i=1}^l \varepsilon_i a_i \equiv r \pmod{p} \quad \varepsilon_i = 0 \text{ or } 1, \text{ not all } \varepsilon_i = 0$$

is solvable for every $r \pmod{p}$.

This theorem is perhaps of some interest in itself and easily implies Grahams conjecture in case each residue occurs with a multiplicity $< \eta_0 p$. To see this observe that if $n_0^{1/10} < \frac{1}{2}$ we can split our set $a_1, \dots, a_p$ into two disjoint sets which satisfy the requirements of Theorem 1 and thus $\sum_{i=1}^p \varepsilon_i$ cannot be unique for $\sum_{i=1}^p \varepsilon_i a_i \equiv 0 \pmod{p}$.

Now we prove Theorem 1. Put $\eta^{1/10} = \delta$. First we prove the following.

Now denote by $F(D)$ the set of all residues of the form $\sum_{x_i \in D} \varepsilon_i x_i$ and with $X + Y = \{x + y; x \in X, y \in Y\}$.

Lemma. *Let $B \subset A$, $|B| > \frac{|A|}{2}$, ($|A| = l > \delta p$). Then there is a $D \subset B$ so that $|F(D)|$ is greater than $\frac{1}{2\delta^2} |D|$.*

To prove the Lemma observe that we can assume that there is a $B_1 \subset B$, $|B_1| > \frac{1}{2}|B|$ that every residue occurs in B_1 with a multiplicity at least $\eta^2 p^{1/2}$. For if not then a simple argument shows that B contains more than $3p^{1/2}$ distinct residues and then by a theorem of Erdős and Heilbronn $\sum_{a_i \in A} \varepsilon_i a_i \equiv r \pmod{p}$ is solvable for all r [1] which contradicts our hypothesis.

Henceforth we only consider B_1 . By the theorem of Dirichlet to every $b \in B_1$ there is an integer $t_b < \frac{1}{\delta^2}$ so that the residue of $t_b \cdot b \pmod{p}$ is an absolute value $< \delta^2 p$. We want to show that there is a $b \in B_1$ for which this $t_b b \pmod{p}$ is an absolute value $> \frac{\delta^3}{8\eta}$. The number of distinct b 's in B_1 is greater than $\frac{\delta}{4\eta}$ (B_1 has at least $\frac{\delta}{4}p$ elements and at most ηp of them are in the same congruence class). Now there are at most $\frac{1}{\delta^2}$ choices for t_b thus there are at most $\frac{1}{\delta^2}$ distinct b 's for which $t_b \cdot b$ is in the same residue class, hence there are at most $\frac{1}{\delta^2} \cdot 2 \frac{\delta^3}{8\eta} = \frac{\delta}{4\eta}$ distinct values of b for which $t_b \cdot b$ is not greater than $\frac{\delta^3}{8\eta}$, but since there are more than $\frac{\delta}{4\eta}$ distinct b 's in B_1 there is a $b \in B_1$ for which

$$(1) \quad \frac{\delta^3}{8\eta} < |t_b \cdot b| < \delta^2 p$$

as stated.

Now we are ready to construct D . We can assume without loss of generality that 1 occurs in B_1 (and is different from the b which we just constructed). Now our set D consists of $t_b \left[\frac{1}{\delta^2} \right] b$'s and $\left[\frac{\delta^3}{8\eta} \right] 1$'s (by our conditions we have at least $\eta^2 p^{1/2}$ 1's and b 's). It easily follows from (1) that the number of sums $\sum \varepsilon_i d_i$, $d_i \in D$ is at least

$$(2) \quad \left[\frac{1}{\delta^2} \right] \left[\frac{\delta^3}{8\eta} \right] > \frac{\delta}{9\eta} > \frac{1}{2\delta^2} \left(t_b \left[\frac{1}{\delta^2} \right] + \left[\frac{\delta^3}{8\eta} \right] \right)$$

(2) follows from $t_b \equiv \left[\frac{1}{\delta^2} \right]$ and $\delta = \eta^{1/10}$ which proves the Lemma.

Unit D from A and apply the Lemma repeatedly. Thus we obtain disjoint sets D_i , $1 \leq i \leq r$ each of which satisfy the Lemma and their union has at least $\frac{|A|}{2}$ elements (since by the Lemma if

$$(3) \quad \left| A - \bigcup_{i=1}^r D_i \right| > \frac{1}{2} |A| \equiv \frac{\delta}{2} p$$

we can select another set D_{r+1}).

Now denote by $F(D_i)$ the set of all residues of the form $\sum_{d_j \in D_i} \varepsilon_i d_i$ by our Lemma

$$(4) \quad |F(D_i)| > |D_i| \frac{1}{2\delta^2}.$$

Now clearly

$$(5) \quad F\left(\bigcup_{i=1}^r D_i\right) = F(D_1) + F(D_2) + \dots + F(D_r).$$

By the Cauchy—Davenport theorem [2]

$$\left|F\left(\bigcup_{i=1}^r D_i\right)\right| \equiv \min\left(p, \sum_{i=1}^r |F(D_i)|\right) = p$$

by (3), (4) and (5), which completes the proof of Theorem 1.

Henceforth we can assume that at least one residue occurs at least $\eta_0 p$ times amongst the a 's. Without loss of generality we can assume that this residue is 1 and that 1 occurs $t \equiv \eta_0 p$ times.

We have to distinguish several cases. First assume $t > \frac{9}{10}p$. Several subcases have to be distinguished. First assume that all a 's are $\equiv p-t$, $1 < a_1 < \dots < a_{p-t} \equiv p-t$. Let $a_r + \dots + a_k \equiv p-t$ be the smallest k with this property, $k < p-t$ is easy to see also $a_1 + \dots + a_{k+1} < p$ is obvious thus

$$a_1 + \dots + a_k + (p - a_1 - a_2 - \dots - a_k)1 \quad \text{and} \quad a_1 + \dots + a_{k+1} + (p - a_1 - \dots - a_{k+1})1$$

give two representations of 0 with different $\sum_{i=1}^p \varepsilon_i$.

Thus at least one of the a 's are $> p-t$. Clearly one cannot have two incongruent a 's in $(p-t, p)$ otherwise $\sum \varepsilon_i$ is clearly not unique. Let $p-t < a_{p-t} < p$. If $a_{p-t} \equiv t$ it must clearly occur with multiplicity one (since otherwise $t > \frac{9}{10}p$ again gives non-uniqueness for $\sum \varepsilon_i$). Observe that in this case $a_1 + \dots + a_{p-t-1} \equiv 2(p-t-1) \equiv p-t$ since $t \equiv p-2$. Let now k be the smallest integer satisfying

$$t > a_1 + \dots + a_k \equiv p-t$$

and now $a_{p-t} + (p - a_{p-t})1$ and $a_1 + \dots + a_k + (p - a_1 - \dots - a_k)1$ give two different values for $\sum_{i=1}^p \varepsilon_i$ what is contradiction.

Thus we can assume $a_{p-t} < t$. But then $a_1 + a_{p-t} < p$ and thus we again get using $p - a_{p-t}$ resp. $p - a_1 - a_{p-t}$ ones two different values of $\sum_{i=1}^p \varepsilon_i$. This disposes the case $t > \frac{9}{10}p$.

Henceforth assume $\eta_0 p < t \leq \frac{9}{10}p$. Again we have to distinguish several cases.

First assume that there are at most $\frac{t}{100}$ residues amongst the a 's greater than

$\frac{t}{100}$. Since there are $p-t$ a 's not congruent 1 there clearly are at least $\frac{p-t}{2}+1$ a 's greater than one but less than $\frac{t}{100}$. Their sum is thus greater than $p-t$. Let $a_1+\dots+\dots+a_r$ the smallest r for which $a_1+\dots+a_r \equiv p-t$ then also $a_1+\dots+a_r+a_{r+1} < p-t+\frac{t}{50} < p$ thus $a_1+\dots+a_r+(p-a_1-\dots-a_r) \cdot 1$ and $a_1+\dots+a_r+a_{r+1}+(p-a_1-\dots-a_r-a_{r+1}) \cdot 1$ again give two different values for $\sum_{i=1}^p \varepsilon_i$.

Henceforth we can assume that there are at least $\frac{t}{100}$ a 's greater than $\frac{t}{100}$ and in fact we can assume that they are all less than $\frac{p-t}{2}$ (since as we proved in the previous case at most one a can be greater than $\frac{p-t}{2}$).

Let now S_1 be a set of $\frac{t}{100}$ a 's which are congruent one and S_2 a disjoint set of $\frac{t}{200}$ a 's which are also congruent one. Let a be one of the residues in $\left(\frac{t}{100}, \frac{p-t}{2}\right)$. Clearly

$$(6) \quad |F(a \cup S_1)| \equiv \frac{2t}{100} = \frac{t}{50} \quad \text{and} \quad |F(A - S_1 - S_2 - a)| >$$

$$|A| - |S_1| - |S_2| - 1 \equiv p - \frac{t}{100} - \frac{t}{200} - 1.$$

Thus by Cauchy-Davenport

$$|F(a \cup S_1 \cup A - S_1 - S_2 - a)| \equiv \min \{p_1 |F(a \cup S_1)| + |F(A - S_1 - S_2 - a)|\} = p.$$

Hence

$$(7) \quad 0 \equiv \alpha_1 \cdot 1 + \sum_i \alpha_{a_i} a_i, \quad \alpha_1 \equiv t - \frac{t}{100}.$$

Now we again have to distinguish two cases. Assume first

$$\sum_i \alpha_{a_i} > \eta_0^2 p \quad (t \equiv \eta_0 p).$$

As stated previously we can assume by the theorem of Erdős and Heilbronn that the number of distinct a 's is less than $3\sqrt{p}$ thus we can assume $\alpha_{a_1} > \frac{\eta_0^2}{3} p^{1/2}$.

Thus by the theorem of Dirichlet there is an $s \equiv \frac{\alpha_{a_1}}{2}$ for which

$$|sa_1| < p^{1/2} \frac{3}{\eta_0^2} < \frac{t}{200}.$$

If $sa_1 > p - \frac{t}{200}$ then $sa_1 + (p - sa_1)1$ and $2sa_1 - (p - 2sa_1)1$ give two representations of D with different $\sum \varepsilon_i$. Thus we can assume $sa_1 < \frac{t}{200}$ but then sa_1 can be replaced by sa_1 ones from S_2 and since $sa_1 \neq s$ this again gives two distinct values of $\sum \varepsilon_i$.

Thus we can assume $\sum \alpha_{a_i} < \eta_0^2 p$. Thus we have at least $p - t - \eta_0^2 p > \frac{p-t}{2}$ a 's distinct from 1 which have not been used in (7). By Erdős—Heilbronn (as used before) at least one of these a 's have a high multiplicity and thus there is an $sa < \frac{t}{100}$. Thus $\alpha_1 < \frac{t}{100}$ since otherwise we could replace sa of the ones by sa and thus we again get two distinct values of $\sum \varepsilon_i$.

Now we omit from A all the a 's occurring in (6) and we obtain a new set A' . Using (6) for A' we again get representation of 0 (7') (we remark that we can assume that α_1 in (7) and α'_1 in (7') are both $\leq t - \frac{t}{100}$ thus we do not run out of ones). Adding the two representations of D we obtain our contradiction.

References

- [1] P. ERDŐS and H. HEILBRONN, On the addition of residue classes mod p , *Acta Arithmetica* **9** (1964), 149—159.
- [2] H. HALBERSTAM and K. F. ROTH, *Sequences*, Oxford, 1969.

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