

Nonoscillation in half-linear differential equations

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Abstract. We establish some necessary conditions on the nonoscillation of the following half-linear second order differential equation

$$[r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \quad t \geq t_0,$$

where $p > 1$ is a constant, $r(t)$ and $c(t)$ are continuous functions from $[t_0, \infty)$ to $[0, \infty)$ with $r(t) > 0$.

1. Introduction

This paper is concerned with the half-linear second order differential equation

$$(E) \quad [r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \quad t \geq t_0,$$

where $p > 1$ is a constant, $r(t)$ and $c(t)$ are continuous functions on $[t_0, \infty)$ for some $t_0 \geq 0$. Throughout the paper, we assume that

- (A₁) $\frac{1}{p} + \frac{1}{q} = 1$;
- (A₂) $r(t) > 0$ for $t \geq t_0$ and $\int_{t_0}^{\infty} r^{1-q}(s)ds = \infty$;
- (A₃) $c(t) \geq 0$ for $t \geq t_0$ and $c(t) \not\equiv 0$ on any interval of the form $[t, \infty)$, $t \geq t_0$.

By a solution of (E) we mean a function $u \in C^1[t_0, \infty)$ such that $r|u'|^{p-2}u' \in C^1[t_0, \infty)$ and that satisfies (E). In [1], ELBERT established the existence, uniqueness and extension to $[t_0, \infty)$ of solutions to the initial value problem for (E). We will say that a nontrivial solution u of (E) is

nonoscillatory if there exists a number $N > 0$ such that $u(t) \neq 0$ for all $t \geq N$. Equation (E) is nonoscillatory if all its solutions are nonoscillatory.

KUSANO, NAITO and OGATA [2], and LI and YEH [3] independently showed that if (E) is nonoscillatory then

$$(1) \quad \int_{t_0}^{\infty} c(s) ds < \infty$$

and

$$(2) \quad \limsup_{t \rightarrow \infty} \pi^{p-1}(t) \int_t^{\infty} c(s) ds \leq 1,$$

where

$$\pi(t) = \int_{t_0}^t r^{1-q}(s) ds, \quad t \geq t_0.$$

It follows from (2) that if (E) is nonoscillatory then

$$(3) \quad \int_{t_0}^{\infty} r^{1-q}(s) \left(\int_s^{\infty} c(\tau) d\tau \right)^q ds < \infty.$$

The purpose of this paper is to improve the results (1), (2), (3), and hence extend the result of LOVELADY [4].

2. Main results

In order to prove our main theorem, we need the following lemma.

Lemma 2.1. *If $u(t)$ is a nonoscillatory solution of (E) which is not eventually a constant, then $u(t)u'(t) > 0$ for all large t .*

PROOF. Without loss of generality, we may assume that $u(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. It follows from (E) that

$$(1) \quad [r(t)|u'(t)|^{p-2}u'(t)]' \leq 0 \quad \text{for } t \geq T_0,$$

which implies that $r(t)|u'(t)|^{p-2}u'(t)$ is nonincreasing on $[T_0, \infty)$. Suppose there exists a $T_1 \geq T_0$ such that $u'(T_1) \leq 0$. Then $r(T_1)|u'(T_1)|^{p-2}u'(T_1) \leq 0$. Since $r(t)|u'(t)|^{p-2}u'(t)$ is decreasing and not identically zero on $[T_0, \infty)$, there exists a $T_2 \geq T_1$ such that

$$r(t)|u'(t)|^{p-2}u'(t) \leq r(T_2)|u'(T_2)|^{p-2}u'(T_2) = -k < 0 \quad \text{for } t \geq T_2,$$

which implies

$$(2) \quad u'(t) \leq -k^{q-1}r^{1-q}(t) \quad \text{for } t \geq T_2.$$

Integrating (2) from T_2 to t , we obtain by (A_2)

$$u(t) \leq u(T_2) - k^{q-1} \int_{T_2}^t r^{1-q}(s) ds \rightarrow -\infty \quad \text{as } t \rightarrow \infty,$$

which contradicts to $u(t) > 0$ on $[T_0, \infty)$. Thus $u'(t) > 0$ on $[T_0, \infty)$. This completes our proof.

Theorem 2.2. *Let*

$$f(t) = \int_t^\infty c(s) ds, \quad t \in [t_0, \infty).$$

If (E) is nonoscillatory, then there exist a number $T_0 \geq t_0$ and a sequence $\{w_k\}(t)_{k=0}^\infty$ of continuous functions from $[T_0, \infty)$ to $(0, \infty)$ with the following properties:

- (a) $w_1 = f$.
- (b) $w_k(t) \leq w_{k+1}(t)$ for $t \geq T_0$ and each integer $k \geq 1$.
- (c) $\int_t^\infty r^{1-q}(s) f^{q-1}(s) w_k(s) ds < \infty$ for $t \geq T_0$ and each integer $k \geq 0$;
and
 $w_{k+1}(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_k(s) ds$ for $t \geq T_0$ and each integer $k \geq 1$.
- (d) If $t \geq T_0$, then $w_0(t) = \lim_{k \rightarrow \infty} w_k(t)$, and the convergence is uniform in each compact subset of $[T_0, \infty)$.
- (e) $\limsup_{t \rightarrow \infty} \pi^{p-1}(t) w_k(t) \leq 1$ for each integer $k \geq 0$.
- (f) $w_0(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_0(s) ds$ for $t \geq T_0$.

PROOF. Let $u(t)$ be a nonoscillatory solution of (E). By Lemma 2.1, without loss of generality, we may assume that $u(t) > 0$ and $u'(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Let

$$w(t) = \frac{r(t)|u'(t)|^{p-2}u'(t)}{|u(t)|^{p-2}u(t)} \quad \text{for } t \geq T_0.$$

Then $w(t) > 0$ and

$$(3) \quad w'(t) = -c(t) - (p-1)r^{1-q}(t)w^q(t) < 0$$

for $t \geq T_0$. This implies that $w(t)$ is decreasing and $\lim_{t \rightarrow \infty} w(t)$ exists. Integrating (3) from t to T , we obtain

$$w(T) - w(t) = - \int_t^T c(s) ds - (p-1) \int_t^T r^{1-q}(s) w^q(s) ds$$

for $T \geq t \geq T_0$. It follows from (1) and the existence of $\lim_{t \rightarrow \infty} w(t)$ that

$$(4) \quad \int_{T_0}^{\infty} r^{1-q}(s) w^q(s) ds < \infty.$$

It follows from (A_2) and the decrease of $w(t)$ that $\lim_{T \rightarrow \infty} w(T) = 0$. This implies

$$(5) \quad w(t) = f(t) + (p-1) \int_t^{\infty} r^{1-q}(s) w^q(s) ds \quad \text{for } t \geq T_0.$$

It is clear from (5) that $w \geq f$ on $[T_0, \infty)$, and hence (4) and (5) imply that

$$(6) \quad \int_{T_0}^{\infty} r^{1-q} f^{q-1}(s) w(s) ds \leq \int_{T_0}^{\infty} r^{1-q}(s) w^q(s) ds < \infty$$

and

$$(7) \quad w(t) \geq f(t) + (p-1) \int_t^{\infty} r^{1-q} f^{q-1}(s) w(s) ds$$

for $t \geq T_0$, respectively. It follows from (E) that $r^{q-1}(t)u'(t)$ is decreasing on $[T_0, \infty)$. Then

$$\begin{aligned} \frac{u(t)}{r^{1-q}(t)u'(t)\pi(t)} &= \frac{u(T_0) + \int_{T_0}^t u'(s) ds}{r^{q-1}(t)u'(t)\pi(t)} \\ &= \frac{u(T_0) + \int_{T_0}^t r^{1-q}(s)r^{q-1}(s)u'(s) ds}{r^{q-1}(t)u'(t)\pi(t)} \\ &\geq \frac{u(T_0) + r^{q-1}(t)u'(t) \int_{T_0}^t r^{1-q}(s) ds}{r^{q-1}(t)u'(t)\pi(t)} \\ &\geq \frac{\pi(t) - \pi(T_0)}{\pi(t)} \end{aligned}$$

for $t \geq T_0$. This implies that

$$\pi^{p-1}(t)w(t) \leq \left(\frac{\pi(t)}{\pi(t) - \pi(T_0)} \right)^{p-1},$$

thus,

$$(8) \quad \limsup_{t \rightarrow \infty} \pi^{p-1}(t)w(t) \leq 1.$$

Let $w_1(t) = f(t)$ on $[T_0, \infty)$, and let

$$w_2(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w_1(s)ds \quad \text{for } t \geq T_0.$$

Then $w_2(t) \geq w_1(t)$ and

$$w_2(t) \leq f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w(s)ds \leq w(t)$$

for $t \geq T_0$. It follows from (8) that $\limsup_{t \rightarrow \infty} \pi^{p-1}(t)w_2(t) \leq 1$. Suppose n is a positive integer and $w_1, w_2, \dots, w_n$ are defined such that $w_1 \leq w_2 \leq \dots \leq w_n \leq w$ on $[T_0, \infty)$, then

$$\int_{T_0}^\infty r^{1-q}(s)f^{q-1}(s)w_k(s)ds < \infty$$

whenever $1 \leq k \leq n$, and

$$w_{k+1}(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w_k(s)ds$$

whenever $1 \leq k \leq n-1$ and $t \geq T_0$. Let w_{n+1} be given by

$$w_{n+1}(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w_n(s)ds.$$

Now

$$\begin{aligned} w_n(t) &= f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w_{n-1}(s)ds \\ &\leq f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w_n(s)ds \\ &\leq f(t) + (p-1) \int_t^\infty r^{1-q}(s)f^{q-1}(s)w(s)ds \\ &\leq w(t), \end{aligned}$$

this implies that $w_n(t) \leq w_{n+1}(t) \leq w(t)$ for $t \geq T_0$. It is clear from (8) that

$$\int_{T_0}^\infty r^{1-q}(s)f^{q-1}(s)w_{n+1}(s)ds \leq \int_{T_0}^\infty r^{1-q}(s)f^{q-1}(s)w(s)ds < \infty.$$

We now see that there is a sequence $\{w_k\}_{k=1}^\infty$ satisfying (a), (b), (c), and

$$(9) \quad w_k(t) \leq w(t)$$

whenever $k \geq 1$ and $t \geq T_0$. Now (8) and (9) give (e). From (c) we see that the family $\{w_1, w_2, \dots\}$ is equicontinuous, so (9) says that there is a subsequence $\{w_{k_j}\}_{j=1}^\infty$ with a locally uniformly limit on $[T_0, \infty)$. This and (b) say that $\{w_k\}_{k=1}^\infty$ has a locally uniform limit, say w_0 , on $[T_0, \infty)$. Clearly, $w_0 \leq w$, so that

$$\int_{T_0}^\infty r^{1-q}(s) f^{q-1}(s) w_0(s) ds < \infty.$$

Now, Lebesgue's Dominated Convergence Theorem yields

$$\int_t^\infty r^{1-q}(s) f^{q-1}(s) w_0(s) ds = \lim_{k \rightarrow \infty} \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_k(s) ds$$

for $t \geq T_0$. This implies (d), and (f) is clear from the above discussion, so that the proof is complete.

Corollary 2.3. *If (E) is nonoscillatory, then*

$$\limsup_{t \rightarrow \infty} \pi^{p-1}(t) \left\{ \int_t^\infty c(s) ds + (p-1) \int_t^\infty r^{1-q}(s) \left(\int_s^\infty c(\tau) d\tau \right)^q ds \right\} \leq 1.$$

PROOF. As in the proof of Theorem 2.2, we have

$$\limsup_{t \rightarrow \infty} \pi^{p-1}(t) w_2(t) \leq 1,$$

and

$$\begin{aligned} w_2(t) &= f(t) + (p-1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_1(s) ds \\ &= f(t) + (p-1) \int_t^\infty r^{1-q}(s) f^q(s) ds, \end{aligned}$$

where $f(t) = \int_t^\infty c(s) ds$. Hence, the proof is complete.

Corollary 2.4. *If (E) is nonoscillatory, then*

$$(10) \quad \int_{t_0}^\infty c(s) \exp \left((p-1) \int_{t_0}^s r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds < \infty$$

and

$$(11) \quad \int_{t_0}^\infty r^{1-q}(s) f^q(s) \exp \left((p-1) \int_{t_0}^s r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds < \infty.$$

PROOF. As in the proof of Theorem 2.2, there is a number $T_0 \geq t_0$ and a function w_0 on $[T_0, \infty)$ such that

$$w_0(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_0(s) ds,$$

where $f(t) = \int_t^\infty c(s) ds$. This implies that

$$(12) \quad w_0'(t) = -c(t) - (p-1)r^{1-q}(t)f^{q-1}(t)w_0(t).$$

Its solution is

$$\begin{aligned} w_0(T_0) - \int_{T_0}^t c(s) \exp \left((p-1) \int_{T_0}^s r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds \\ = w_0(t) \exp \left((p-1) \int_{T_0}^t r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) > 0. \end{aligned}$$

Hence,

$$w_0(T_0) > \int_{T_0}^t c(s) \exp \left((p-1) \int_{T_0}^s r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds.$$

This implies

$$(13) \quad \int_{T_0}^\infty c(s) \exp \left((p-1) \int_{T_0}^s r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds < \infty.$$

Clearly, (13) is equivalent to (10). Let z be given on $[T_0, \infty)$ by

$$z(t) = \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_0(s) ds.$$

Then

$$z'(t) = -r^{1-q}(t) f^{q-1}(t) w_0(t) = -r^{1-q}(t) f^q(t) - (p-1)r^{1-q}(t) f^{q-1}(t) z(t),$$

which implies that (11) holds. Hence, the proof is complete.

References

- [1] Á. ELBERT, A half-linear second order differential equations, 30: Qualitative Theory of Differential Equations, *Colloquia Math. Soc. Janos Bolyai, Szeged*, 1979, pp. 153–180.
- [2] T. KUSANO, Y. NAITO and A. OGATA, Strong oscillation and nonoscillation of quasilinear differential equations of second order, *Differential Equations and Dynamical Systems* **2** (1994), 1–10.
- [3] H. J. LI and C. C. YEH, Oscillation criteria for nonlinear differential equations, *Houston J. Math.*, **21** (1995), 801–811.

- [4] D. L. LOVELADY, Nonoscillation in linear second order ordinary differential equations, *Hiroshima Math. J.* **5** (1975), 135–139.

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