

A new algebra of distributions; initial value problems involving Schwartz distributions II.

By GREGERS KRABBE (West Lafayette, Ind.)

§ 3. The operational calculus

Let $\mathcal{C}^\infty$ be the linear space of all the complex-valued functions which are infinitely differentiable on $\mathbf{R}$. Further, let W be the linear space of all the functions $w(\cdot)$ in $\mathcal{C}^\infty$ such that $w^{(k)}(0)=0$ for each integer $k \geq 0$. For example, if $q(\cdot)$ is the function defined by $q(0)=0$ and

$$(3.1) \quad q(x) = \exp\left(\frac{-1}{|x|}\right) \quad (\text{for } x \neq 0),$$

then

$$(3.2) \quad q(\cdot) \text{ belongs to } W.$$

An operator is a linear mapping of W into W . If A is an operator and if $w(\cdot) \in W$, we denote by $.Aw(\cdot)$ the function that the operator A assigns to $w(\cdot)$.

3.3. *The space $\mathcal{A}$.* Let $\mathcal{A}$ be the space (denoted $\mathcal{A}_\omega$ in [7]) of all the operators A such that the equation

$$(3.4) \quad .A(w_1 \wedge w_2)(\cdot) = (.Aw_1) \wedge w_2(\cdot)$$

holds whenever $w_1(\cdot)$ and $w_2(\cdot)$ belong to W . The operation $\wedge$ was defined in 2.33.

The linear space $\mathcal{A}$ is a subalgebra of the algebra of operators, multiplication being defined in the usual way: the product $A_1 A_2$ of two operators is determined by the equation

$$.A_1 A_2 w(\cdot) = .A_1(.A_2 w)(\cdot) \quad (\text{for } w(\cdot) \text{ in } W).$$

In fact $\mathcal{A}$ is a commutative algebra [6, 1.38] and $\mathcal{A}$ contains the differentiation operator D defined by

$$(3.5) \quad .Dw(\cdot) = w'(\cdot) \quad (\text{for } w(\cdot) \text{ in } W);$$

see [6, 1.30]. Let I be the identity-operator defined by $.Iw(\cdot) = w(\cdot)$ for every $w(\cdot)$ in W : the operator I is the multiplicative unit of the algebra $\mathcal{A}$.

If $f(\cdot) \in \mathcal{F}$ we denote by f the mapping defined by

$$(3.6) \quad .fw(\cdot) = f \wedge w'(\cdot) \quad (\text{for } w(\cdot) \text{ in } W);$$

it can be proved that f is an operator, called *the operator of the function $f(\cdot)$* ; in fact, $f \in \mathcal{A}$ (see 1.34 in [6]). The identity-operator I is the operator 1 of the unit constant function $1(\cdot)$ defined by $1(t)=1$ for $t \in \mathbf{R}$ (see [6, 1.29]); thus,

$$(3.7) \quad 1A = A = A1 \quad (\text{for } A \text{ in } \mathcal{A}).$$

Moreover, the transformation $f(\cdot) \mapsto f$ is a linear injection of $\mathcal{F}$ into $\mathcal{A}$ (see 1.34 in [6]).

3.8. *Reorientation.* We shall define a linear injection $R \mapsto R^1$ of $\mathcal{B}$ into $\mathcal{A}$ such that

$$(3.9) \quad [f]^0 = f \quad (\text{when } f(\cdot) \in \mathcal{F}),$$

$$(3.10) \quad [F \otimes S]^1 = F^1 D^{-1} S^1 \quad (\text{for } F \text{ and } S \text{ in } \mathcal{B}),$$

$$(3.11) \quad ([1]^0 \otimes S)^1 = D^{-1} S^1 \quad (\text{for } S \text{ in } \mathcal{B}),$$

and

$$(3.12) \quad \delta^1 = D1_+.$$

We begin with some preliminary definitions and lemmas.

3.13. **Lemma.** *If $w(\cdot) \in W$ then both $w_-(\cdot)$ and $w_+(\cdot)$ belong to $\mathcal{C}^\infty$.*

PROOF. Let k be any integer ≥ 1 . Since

$$w^{(k)}(0) = \lim_{t \rightarrow 0} \frac{w^{(k-1)}(t) - w^{(k-1)}(0)}{t - 0},$$

the equation

$$(2) \quad 0 = \lim_{t \rightarrow 0} \frac{w^{(k-1)}(t)}{t}$$

is an immediate consequence of the fact that $w^{(n)}(0)=0$ for each integer $n \geq 0$. Set $f_1(\cdot)=w_-(\cdot)$ and $f_2(\cdot)=w_+(\cdot)$ (see 2.34—2.35). Suppose that $1 \leq m \leq 2$: we must prove that $f_m^{(k)}(t)$ exists for any t in $\mathbf{R}$. Since $f_m^{(k)}(t)$ exists for $t \neq 0$, it will suffice to prove that $f_m^{(k)}(0)=0$. Since $f_m^{(0)}(0)=0$, we proceed by induction on k . To that effect, assume that $f_m^{(k-1)}(0)=0$, which implies that

$$(3) \quad f_m^{(k)}(0) = \lim_{t \rightarrow 0} \frac{f_m^{(k-1)}(t) - 0}{t} = 0;$$

the second equation is obtained by noting that $f_m^{(k-1)}(t)/t$ either equals 0 or it equals $w^{(k-1)}(t)/t$ (and using (2)). This completes the induction proof of the equation $f_m^{(k)}(0)=0$ for each integer k .

3.14. **Definition.** Let $\mathcal{C}_-^\infty$ (respectively, $\mathcal{C}_+^\infty$) be the linear space of all the functions in $\mathcal{C}^\infty$ which vanish on $[0, \infty)$ (respectively, on $(-\infty, 0]$).

3.15. **Lemma.** *If $w(\cdot) \in W$ then $w_-(\cdot) \in \mathcal{C}_-^\infty$ and $w_+(\cdot) \in \mathcal{C}_+^\infty$; moreover, $W = \mathcal{C}_-^\infty + \mathcal{C}_+^\infty$.*

PROOF. The two properties $w_-(\cdot) \in \mathcal{C}_-^\infty$ and $w_+(\cdot) \in \mathcal{C}_+^\infty$ are immediate from 3.13 and 2.34—2.35. Since both $\mathcal{C}_-^\infty$ and $\mathcal{C}_+^\infty$ are subsets of W , we only need to prove that W is a subset of $\mathcal{C}_-^\infty + \mathcal{C}_+^\infty$. To that effect, take $w(\cdot)$ in W and note that $w(\cdot) = w_-(\cdot) + w_+(\cdot)$ (by 2.34—2.35): the conclusion $w(\cdot) \in \mathcal{C}_-^\infty + \mathcal{C}_+^\infty$ is now immediate from the fact that $w_-(\cdot) \in \mathcal{C}_-^\infty$ and $w_+(\cdot) \in \mathcal{C}_+^\infty$.

3.16. **Lemma.** *There is a bilinear mapping*

$$(3.17) \quad (\alpha(\cdot), T) \mapsto \alpha \square T(\cdot) \in \mathcal{C}_+^\infty$$

of the cartesian product $\mathcal{C}_+^\infty \times \mathfrak{B}_+$ into $\mathcal{C}_+^\infty$ such that

$$(3.18) \quad [\alpha \square T]^0 = [\alpha]^0 * T.$$

PROOF. Suppose that $\alpha(\cdot) \in \mathcal{C}_+^\infty$. If $x \in \mathbf{R}$, let $\alpha_x(\cdot)$ be the function defined by $\alpha_x(t) = \alpha(x-t)$ (for $t \in \mathbf{R}$). If $T \in \mathfrak{B}_+$ we denote by $\alpha \square T(\cdot)$ the function defined by

$$(4) \quad \alpha \square T(x) = T(\alpha_x) \quad (\text{for } x \in \mathbf{R}).$$

Since the supports of $[\alpha]^0$ and T are both contained in $[0, \infty)$, these distributions satisfy condition (Σ') in [4, p. 383]; we may therefore apply Proposition 1 in [4, p. 402] with $s = \infty = t$: what is there called E^s and E^{s-t} becomes $\mathcal{C}^\infty$ (see [4, p. 440]); further, D'^s becomes D' ; noting that $\alpha * T$ is what we denote by $[\alpha]^0 * T$, Proposition 1 states that $[\alpha]^0 * T(x)$ equals the righthand side of (4): consequently, $[\alpha \square T]^0 = [\alpha]^0 * T$. Having thus verified 3.18, we can use the second assertion in Proposition 1 [loc cit.] to state that

$$(5) \quad \alpha \square T(\cdot) \text{ belongs to } \mathcal{C}^\infty.$$

Since both $\mathbf{O}([\alpha]^0)$ and $\mathbf{O}(T)$ contain the interval $(-\infty, 0)$, it follows from 2.8 that

$$\mathbf{O}([\alpha]^0 * T) \supset (-\infty, 0),$$

whence the relation $\mathbf{O}(\alpha \square T) \supset (-\infty, 0)$ now follows from 3.18: combining with (5), we conclude that $\alpha \square T(\cdot)$ belongs to $\mathcal{C}_+^\infty$.

In consequence, we have established that the mapping 3.17 is into $\mathcal{C}_+^\infty$; it is readily inferred from 3.18 that it is a bilinear mapping.

3.19. **Theorem.** *There is a bilinear mapping*

$$(3.20) \quad (R, w(\cdot)) \mapsto R \triangle w(\cdot) \in W$$

of the cartesian product $\mathfrak{B} \times W$ into W such that

$$(3.21) \quad [R \triangle w]^0 = R \otimes [w]^0 \quad (\text{for } w(\cdot) \text{ in } W),$$

and

$$(3.22) \quad R \triangle q(\cdot) = 0 \text{ implies } R = 0,$$

where $q(\cdot)$ is the function defined by 3.1.

PROOF. Suppose that $R \in \mathfrak{B}$. If $w(\cdot) \in W$ it follows from 3.13 that $w_+(\cdot) \in \mathcal{C}_+^\infty$; we may therefore set $\alpha = w_+$ in 3.16 to infer that

$$(1) \quad w_+ \square R_+(\cdot) \text{ belongs to } \mathcal{C}_+^\infty \quad (\text{since } R_+ \in \mathfrak{B}_+),$$

and

$$(2) \quad [w_+ \square R_+]^0 = R_+ * [w_+]^0;$$

the last equation is from 3.18 and 2.9.

Since $w_+^\vee(x) = w_-(-x) = 0$ when $-x > 0$, we see that $w_-^\vee(\cdot)$ vanishes on $(-\infty, 0)$; this function $w_-^\vee(\cdot)$ therefore belongs to $\mathcal{C}_+^\infty$. On the other hand, $R_- \in \mathfrak{B}_-$, so that $\mathbf{O}(R_-) \supset (0, \infty)$, whence

$$\mathbf{O}(R_-)^\vee \supset (-\infty, 0) \quad (\text{by 2.6}).$$

Thus, $w_-^\vee(\cdot) \in \mathcal{C}_+^\infty$ and $R_-^\vee \in \mathfrak{B}_+$; from 3.16 it therefore follows that

$$(3) \quad w_-^\vee \square R_-^\vee(\cdot) \text{ belongs to } \mathcal{C}_+^\infty$$

and

$$(4) \quad [w_-^\vee \square R_-^\vee]^0 = R_-^\vee * [w_-^\vee]^0;$$

the last equation is from 3.18 and 2.9. From (3) it follows easily that

$$(5) \quad (w_-^\vee \square R_-^\vee)^\vee \text{ belongs to } \mathcal{C}_-^\infty.$$

We now define the function $R \triangle w(\cdot)$ by the equation

$$(6) \quad R \triangle w(\cdot) = -(w_-^\vee \square R_-^\vee)^\vee(\cdot) + w_+ \square R_+(\cdot).$$

From (1), (5), (6), and 3.15 it follows that $R \triangle w(\cdot)$ belongs to W ; it is now easily verified that the mapping 3.21 is bilinear. It remains to prove 3.21—3.22. In view of 2.3, Equation (6) implies that

$$[R \triangle w]^0 = -[w_-^\vee \square R_-^\vee]^{0\vee} + [w_+ \square R_+]^0,$$

so that (4) and (2) give

$$(7) \quad [R \triangle w]^0 = -(R_-^\vee * [w_-^\vee]^0)^\vee + R_+ * [w_+]^0.$$

From (7), 2.3, and 2.12 we see that

$$[R \triangle w]^0 = -R_- * [w_-]^0 + R_+ * [w_+]^0;$$

Conclusion 3.21 now comes from 2.18 and 1.28. Finally, suppose that $R \triangle q(\cdot) = 0$; therefore,

$$(8) \quad 0 = (R \triangle q)_+(\cdot) = q_+ \square R_+(\cdot)$$

and

$$(9) \quad 0 = (R \triangle q)_-^\vee(\cdot) = (q_-^\vee \square R_-^\vee)(\cdot);$$

the right-hand equations are immediate from (6), (5), and (1)—(2). From (8)—(9) and 3.18 we see that

$$(10) \quad [q_+]^0 * R_+ = \mathbf{0} = [q_-^\vee]^0 * R_-^\vee;$$

but $[q_+]^0$ and $[q_-]^0$ belong to the Schwartz space D'_+ , which has the property that $A * T = 0$ implies $T = 0$ whenever $A \neq 0$ (see [8, pp. 172—173]); since $[q_+]^0 \neq 0$ and $[q_-]^0 \neq 0$, the conclusion $R_+ = 0 = R_-$ now come from (10): since $R_- = 0$ implies $R_+ = 0$ (by 2.4), we have $R_- = 0 = R_+$, whence our conclusion $R = 0$.

3.23. Definition. If $R \in \mathfrak{B}$ we denote by R^Δ the mapping that assigns to each $w(\cdot)$ in W the function $R \Delta w(\cdot)$.

3.24. Remark. If $w(\cdot) \in W$ then $R \Delta w \in W$ (by 3.20); therefore, R^Δ is the operator defined by

$$(3.25) \quad .R^\Delta w(\cdot) = R \Delta w(\cdot) \quad (\text{for } w(\cdot) \text{ in } W).$$

From 3.21 we see that

$$(3.26) \quad [.R^\Delta w]^0 = R \otimes [w]^0 \quad (\text{for } w(\cdot) \text{ in } W).$$

3.27. Lemma. If $R \in \mathfrak{B}$ then $R^\Delta \in \mathcal{A}$.

PROOF. In view of 3.24, it only remains to verify that 3.4 holds in case $A = R^\Delta$. The equations

$$(1) \quad [.R^\Delta (w_1 \wedge w_2)]^0 = R \otimes [w_1 \wedge w_2]^0 = R \otimes ([w_1]^0 \otimes [w_2]^0)$$

are from 3.26 and 2.32. From (1) and 2.28 it follows that

$$(2) \quad [.R^\Delta (w_1 \wedge w_2)]^0 = (R \otimes [w_1]^0) \otimes [w_2]^0 = [.R^\Delta w_1]^0 \otimes [w_2]^0;$$

the last equation is from 3.26. From (2) and 2.32 we see that

$$[.R^\Delta (w_1 \wedge w_2)]^0 = [(R^\Delta w_1) \wedge w_2]^0;$$

in view of 1.11, the conclusion

$$.R^\Delta (w_1 \wedge w_2)(\cdot) = (.R^\Delta w_1) \wedge w_2(\cdot)$$

is now at hand.

3.28. Lemma. If F and S belong to $\mathfrak{B}$, then

$$(3.29) \quad (F \otimes S)^\Delta = F^\Delta S^\Delta.$$

PROOF. For any $w(\cdot)$ in W it follows from 3.26 that

$$[.(F \otimes S)^\Delta w]^0 = (F \otimes S) \otimes [w]^0 = F \otimes (S \otimes [w]^0)$$

consequently, two more applications of 3.26 give

$$(3) \quad [.(F \otimes S)^\Delta w]^0 = F \otimes [.S^\Delta w]^0 = [.F^\Delta (.S^\Delta w)]^0.$$

From (3) and 1.11 we therefore have

$$.(F \otimes S)^\Delta w(\cdot) = .F^\Delta (.S^\Delta w)(\cdot) = .(F^\Delta S^\Delta)w(\cdot);$$

the last equation is from the definition of multiplication of operators. Since $w(\cdot)$ is an arbitrary element of W , the proof is completed.

3.30. Definition. If $R \in \mathfrak{B}$, we set $R^1 = R^\Delta D$.

3.31. *Remark.* Since D and R^Δ belong to the algebra $\mathcal{A}$ (see 3.3 and 3.27), we see that $R^1 \in \mathcal{A}$. Let us verify that

$$(3.32) \quad [R^1 w]^0 = R \otimes [w']^0 \quad (\text{for } w(\cdot) \in W):$$

indeed, the equations

$$[R^1 w]^0 = [R^\Delta (\cdot Dw)]^0 = [R^\Delta w']^0 = R \otimes [w']^0$$

are from 3.30 and 3.26.

3.33. **Theorem.** *If F and S belong to $\mathfrak{B}$, then*

$$(3.34) \quad [F \otimes S]^1 = F^1 D^{-1} S^1.$$

PROOF. The equations

$$[F \otimes S]^1 = [F \otimes S]^\Delta D = (F^\Delta S^\Delta) D = (F^\Delta D) D^{-1} (S^\Delta D)$$

are from 3.30, 3.29, and the associativity of the algebra $\mathcal{A}$; another application of 3.30 now gives 3.34.

3.35. **Theorem.** *If $f(\cdot) \in \mathcal{F}$ then*

$$(3.36) \quad [f]^{01} = f.$$

PROOF. If $w(\cdot) \in W$ the equations

$$[f]^{01} w]^0 = [f]^0 \otimes [w']^0 = [f \wedge w']^0 = [fw]^0$$

are from 3.32, 2.32, and 3.6. From 1.11 it therefore follows that $[f]^{01} w(\cdot) = fw(\cdot)$; since $w(\cdot)$ is arbitrary, Conclusion 3.36 is at hand.

3.37. *Remarks.* If $f(\cdot) \in \mathcal{F}$ we can combine 3.36 with 3.34 to obtain

$$(3.38) \quad ([f]^0 \otimes S)^1 = f D^{-1} S^1 \quad (\text{if } S \in \mathfrak{B}).$$

Setting $f(\cdot) = 1(\cdot)$ in 3.38, we obtain

$$(3.39) \quad ([1]^0 \otimes S)^1 = 1 D^{-1} S^1 = D^{-1} S^1;$$

the second equation is from 3.7. Setting $S = \delta$ in 3.39, we obtain $([1]^0 \otimes \delta)^1 = D^{-1} \delta^1$; consequently, the equations

$$(4) \quad \delta^1 = D([1]^0 \otimes \delta)^1 = D(\delta \otimes [1]^0)^1 = D[1]_+^{01}$$

are from 2.27 and 2.29. From (4), 1.28, and 3.36 it follows that

$$(3.40) \quad \delta^1 = D 1_+.$$

3.41. **Theorem.** *The transformation $R \rightarrow R^1$ is a linear injection of $\mathfrak{B}$ into $\mathcal{A}$. In particular,*

$$(3.42) \quad T^1 = S^1 \text{ implies } T = S.$$

PROOF. The linearity follows directly from Definitions 3.30 and 3.25 (the bilinearity property is stated in 3.19); the fact that $R^1 \in \mathcal{A}$ was verified in 3.31. To prove 3.42, set $R = T - S$: the hypothesis $R^1 = 0$ implies $R^\Delta D = 0$ (by 3.30); right-multiplying by D^{-1} both sides of this equation, we obtain $R^\Delta = 0$, whence $\cdot R^\Delta q(\cdot) = 0$ (since $q(\cdot) \in W$: see 3.2), whence $R^\Delta q(\cdot) = 0$ (by 3.25), whence the conclusion $R = 0$ now comes from 3.22.

3.43. *Reorientation.* We have now proved all the properties announced in 3.9–3.12. We now prepare for § 4. If $k \geq 0$ we denote by $Y_{k+1}(\cdot)$ the function defined by

$$(3.44) \quad Y_{k+1}(t) = \frac{t^k}{k!} \quad (\text{for } t \in \mathbb{R}).$$

It is not hard to prove that

$$(3.45) \quad Y_{k+1} = \frac{D}{D^{k+1}} \quad (\text{see 4.5 in [7]}).$$

3.46. **Lemma.** *If $G \in \mathfrak{B}$ and $n \geq 1$ then $\partial^n([Y_n]^0 \otimes G) = G$.*

PROOF. Since $Y_1(\cdot) = 1(\cdot)$, we can use 2.31 to obtain

$$(1) \quad \partial([Y_1]^0 \otimes G) = G.$$

Next, observe that the equations

$$[1]^0 \otimes ([Y_n]^0 \otimes G) = ([1]^0 \otimes [Y_n]) \otimes G = [1 \wedge Y_n] \otimes G$$

come from 2.28 and 2.32; since $1 \wedge Y_n(\cdot) = Y_{n+1}(\cdot)$ (see 2.33 and 3.44), we have

$$(2) \quad [1]^0 \otimes [Y_n]^0 \otimes G = [Y_{n+1}]^0 \otimes G.$$

The equations

$$(3) \quad \partial^{n+1}([Y_{n+1}]^0 \otimes G) = \partial^{n+1}([1]^0 \otimes [Y_n]^0 \otimes G) = \partial^n([Y_n]^0 \otimes G)$$

are from (2) and 2.31 (with $S = [Y_n]^0 \otimes G$). The property $\partial^k([Y_k]^0 \otimes G) = G$ holds for $k=1$ (by (1)); it holds for $k=n+1$ whenever it holds for $k=n$ (by (3)); therefore, it holds for every integer $k \geq 1$.

§ 4. Initial values of distributions

The following notation and terminology was introduced in [6]. 4.1.

Suppose that $a < 0$ and $A \in \mathcal{A}$ (see 3.3). We say that A agrees with B on the interval $(a, 0)$ if $B \in \mathcal{A}$ and

$$\cdot A w(t) = \cdot B w(t) \quad (\text{for } a < t < 0 \text{ and any } w(\cdot) \text{ in } W).$$

The relation $A \subset B$ means that there exists some number $a < 0$ such that A agrees with B on the interval $(a, 0)$.

4.2. *The space $\mathcal{B}_\omega$.* As in [7], we denote by $\mathcal{B}_\omega$ the family of all the elements B of $\mathcal{A}$ such that the relation $f \subset B$ holds for some $f(\cdot)$ in $\mathcal{F}$. Recall that f is the operator of the function $f(\cdot)$ (see 3.6).

4.3. *Remark.* In consequence of 4.1—4.2, we can say that $B \in \mathcal{B}_\omega$ if (and only if) $B \in \mathcal{A}$ and there exists a function $f(\cdot)$ in $\mathcal{F}$ and a number $a < 0$ such that

$$.Bw(t) = .fw(t) \quad (\text{for } a < t < 0 \text{ and } w(\cdot) \in W).$$

4.4. *Derivable operators.* An operator B is said to be *derivable* if $B \in \mathcal{A}$ and if the relation $f \subset B$ holds for some $f(\cdot)$ in $\mathcal{F}$ such that $|f(0_-)| < \infty$. If B is derivable, there exists a unique complex number $\langle B, 0_- \rangle$ such that the equation $\langle B, 0_- \rangle = f(0_-)$ holds for some function $f(\cdot)$ in $\mathcal{F}$ such that $f \subset B$ (see [6, 5.0]). We set

$$(4.5) \quad \partial_t B = BD - \langle B, 0_- \rangle D.$$

We define $\partial_t^n B$ recursively by the equations $\partial_t^0 B = B$ and $\partial_t^{k+1} B = \partial_t(\partial_t^k B)$.

4.6. *Remark.* If $f(\cdot) \in \mathcal{F}$ and if $|f(0_-)| < \infty$ then $\langle f, 0_- \rangle = f(0_-)$: see 2.17 in [7].

4.7. *Reorientation.* One of our aims is to prove that the transformation $R \rightarrow R^1$ is an injection of $\mathfrak{B}$ into $\mathcal{B}_\omega$. Our main result will be obtained by relating distributional derivation $R \mapsto \partial R$ to the operation $B \mapsto \partial_t B$ (on operators): as we shall see, if R is an arbitrary distribution such that $\partial R \in \mathfrak{B}$, then $R \in \mathfrak{B}$, the operator R^1 is derivable, and $(\partial R)^1 = \partial_t R^1$; moreover, it is natural to consider $\langle R^1, 0_- \rangle$ as the *initial value* (denoted $R(0_-)$ in the Introduction) of the distribution R .

4.8. **Lemma.** If $X \in \mathcal{B}_\omega$ then X/D is derivable, and $\langle \frac{X}{D}, 0_- \rangle = 0$.

PROOF. Set $G=1$ in [7, 3.5].

4.9. **Theorem.** If $F \in \mathfrak{B}$ then $F^1 \in \mathcal{B}_\omega$.

PROOF. From 1.20 and 1.18 it follows the existence of a function $f(\cdot)$ in $\mathcal{F}$ and a distribution L in $(\mathcal{L})$ such that

$$(1) \quad F^1 = [f]^{01} + L^1 + F_+^1.$$

We intend to prove that $f \subset F^1$. Take any $w(\cdot)$ in W . From (1) and 3.36 it follows that

$$(2) \quad .F^1 w(\cdot) = .fw(\cdot) + (.L^1 w(\cdot) + .F_+^1 w(\cdot)).$$

On the other hand, 3.32 gives

$$(3) \quad [.L^1 w + .F_+^1 w]^0 = L \otimes [w']^0 + F_+ \otimes [w']^0.$$

Since $L \in (\mathcal{L})$ it follows from 2.24 that $L \otimes [w']^0$ belongs to $(\mathcal{L})$, so that 1.16 therefore insures the existence of a number $a < 0$ such that

$$(4) \quad \bigcirc (L \otimes [w']^0) \supset (a, \infty).$$

Next, observe that $F_+ \otimes [w']^0$ belongs to $\mathfrak{B}_+$ (by 2.23), so that

$$(5) \quad \bigcirc (F_+ \otimes [w']^0) \supset (-\infty, 0) \quad (\text{by 1.13}).$$

Combining (3), (4), and (5), we can use 1.6 to obtain

$$\mathbf{0} ([.L_1 w + .F_+^1 w]^0) \supset (a, \infty) \cap (-\infty, 0) = (a, 0).$$

Thus, $[.L_1 w + .F_+^1 w]^0$ equals $\mathbf{0}$ in $(a, 0)$, whence $.L_1 w(\cdot) + F_+^1 w(\cdot) = 0$ on the interval $(a, 0)$ (by 1.11); since the functions are continuous we therefore have

$$.L_1 w(t) + .F_+^1 w(t) = 0 \quad (\text{for } a < t < 0);$$

consequently, (2) gives

$$(6) \quad .F_+^1 w(t) = .f w(t) \quad (\text{for } a < t < 0).$$

Since $w(\cdot)$ is an arbitrary element of W , Equation (6) states that the operator f agrees with F^1 on the interval $(a, 0)$: therefore, $f \subset F^1$. The conclusion $F^1 \in \mathcal{B}_\omega$ is now immediate from Definition 4.2.

4.10. Theorem. *If R is a distribution such that ∂R belongs to $\mathfrak{B}$, then R also belongs to $\mathfrak{B}$, the operator R^1 is derivable, and $\partial_t R^1 = (\partial R)^1$.*

PROOF. In view of our hypothesis $\partial R \in \mathfrak{B}$, we can set $S = \partial R$ in 2.31 to obtain

$$\partial([1]^0 \otimes \partial R) = \partial R;$$

thus, both $[1]^0 \otimes \partial R$ and R have the same derivative: therefore, they differ by a constant function [4, p. 328]; thus, there exists a number c such that

$$(1) \quad R = [f_c]^0 + [1]^0 \otimes \partial R,$$

where $f_c(\cdot)$ is the constant function $f_c(\cdot) = c$. Since ∂R belongs to $\mathfrak{B}$ (by hypothesis), it follows from (1) and 2.20 that $R \in \mathfrak{B}$ (recall 1.27). From (1) and 3.39 it results that

$$(2) \quad R^1 = [f_c]^{01} + \frac{(\partial R)^1}{D} = f_c + \frac{(\partial R)^1}{D};$$

the second equation is from 3.36. Since f_c is the operator of the constant function $f_c(\cdot) = c$, it follows from 4.6 that

$$(3) \quad \langle f_c, 0_- \rangle = c.$$

Note that $(\partial R)^1$ belongs to $\mathcal{B}_\omega$ (by 4.9 and our hypothesis $\partial R \in \mathfrak{B}$); we may therefore set $X = (\partial R)^1$ in 4.8 to obtain

$$(4) \quad \left\langle \frac{(\partial R)^1}{D}, 0_- \right\rangle = 0.$$

From (2), (3), and (4) we see that

$$\langle R^1, 0_- \rangle = \langle f_c, 0_- \rangle + \left\langle \frac{(\partial R)^1}{D}, 0_- \right\rangle = c + 0 = c.$$

Therefore, 4.5 gives

$$\partial_t R^1 = D R^1 - c D = D \left(f_c + \frac{(\partial R)^1}{D} \right) - f_c D = (\partial R)^1;$$

the middle equation is from (2). Thus, the conclusion $\partial_t R^1 = (\partial R)^1$ is at hand.

4.11. **Lemma.** *Let F be a distribution. The implication*

$$(5) \quad \partial^v F \in \mathfrak{B} \quad \text{implies} \quad \partial_t^v F^1 = (\partial^v F)^1$$

holds for every integer $v \geq 0$.

PROOF. Clearly, (5) holds for $v=0$. To proceed by induction, assume that (5) holds for $v=k-1$:

$$(6) \quad \partial^{k-1} F \in \mathfrak{B} \quad \text{implies} \quad \partial_t^{k-1} F^1 = (\partial^{k-1} F)^1.$$

Let us prove that (5) holds for $v=k$. If $\partial^k F \in \mathfrak{B}$ then $\partial(\partial^{k-1} F) \in \mathfrak{B}$, so that 4.10 gives

$$(7) \quad \partial^{k-1} F \quad \text{belongs to} \quad \mathfrak{B}$$

and

$$(8) \quad \partial_t(\partial^{k-1} F)^1 = [\partial(\partial^{k-1} F)]^1.$$

From (7) and (6) we see that

$$(9) \quad \partial_t(\partial_t^{k-1} F^1) = \partial_t(\partial^{k-1} F)^1:$$

combining (9) with (8), we obtain $\partial_t^k F^1 = [\partial(\partial^{k-1} F)]^1$. Thus, (5) holds for $v=k$ when it holds for $v=k-1$.

4.12. **Lemma.** *Let F be a distribution. If m is an integer ≥ 1 such that $\partial^m F \in \mathfrak{B}$, then*

$$(4.13) \quad \partial^v F \in \mathfrak{B} \quad \text{and} \quad \partial_t^v F^1 = (\partial^v F)^1 \quad \text{for } 0 \leq v \leq m.$$

PROOF. First, suppose that $0 \leq v \leq m-1$. Let us prove that

$$(10) \quad \partial^{v+1} F \in \mathfrak{B} \quad \text{implies} \quad \partial^v F \in \mathfrak{B}.$$

To that effect, observe that the hypothesis $(\partial^{v+1} F \in \mathfrak{B})$ gives $\partial(\partial^v F) \in \mathfrak{B}$, whence the conclusion $\partial^v F \in \mathfrak{B}$ follows immediately

4.10. Consequently, we have proved that

$$(11) \quad 0 \leq v \leq m-1 \quad \text{and} \quad \partial^v F \notin \mathfrak{B} \quad \text{implies} \quad \partial^{v+1} F \notin \mathfrak{B}.$$

Next, let M be the set of all the non-negative integers $v \leq m$ such that $\partial^v F \notin \mathfrak{B}$: since $\partial^m F \in \mathfrak{B}$ (by hypothesis), we have $m \notin M$, whence

$$(12) \quad v \in M \quad \text{implies} \quad 0 \leq v \leq m-1.$$

In view of 4.11, our conclusion 4.13 can be obtained by proving that

$$(13) \quad 0 \leq v \leq m \quad \text{implies} \quad \partial^v F \in \mathfrak{B}.$$

We shall prove (13) by contradiction: if (13) fails, then M is a non-void subset of the positive integers: it therefore follows (from (12)) the existence of an integer

$$(14) \quad v = \max M \quad (\text{see [3, p. 64]}).$$

Since $v \in M$, it follows from (12) that $0 \leq v \leq m-1$. Since $v \in M$, we have $\partial^v F \notin \mathfrak{B}$: from (11) it now follows that $\partial^{v+1} F \notin \mathfrak{B}$. Consequently, $0 \leq v+1 \leq m$ and $\partial^{v+1} F \notin \mathfrak{B}$: therefore, $v+1$ belongs to M , which contradicts (14). This contradiction establishes (13); from (13) and 4.11 now follows 4.13.

§ 5. The main result

Throughout this section, μ is a polynomial μ_k ($k=0, 1, 2, \dots$) of degree $m \geq 1$ (that is, μ is a sequence of complex numbers such that $\mu_m \neq 0$ and $\mu_k = 0$ for $k > m$). As usual,

$$(5.1) \quad \mu(D) = \mu_0 + \mu_1 D + \dots + \mu_m D^m.$$

If u is a distribution we set

$$(5.2) \quad \mu(\partial)u = \mu_0 u + \mu_1 \partial u + \dots + \mu_m \partial^m u.$$

If λ is a polynomial whose degree is smaller than m , we denote by $g^\lambda(\cdot)$ the unique continuous function $h_m(\cdot)$ such that $D\lambda(D)/\mu(D) = h_m$ (see [7, 5.4]):

$$(5.3) \quad D \frac{\lambda(D)}{\mu(D)} = g^\lambda_\mu.$$

5.4 *Remark.* In the particular case where λ is the polynomial such that $\lambda(D) = 1$, we set $g_\mu = g^1_\mu$, so that

$$(5.5) \quad \frac{D}{\mu(D)} = g_\mu.$$

If $S \in \mathfrak{B}$ then

$$(5.6) \quad \frac{S^1}{\mu(D)} = [[g_\mu]^0 \otimes S]^1.$$

To prove 5.6 it suffices to note the equations

$$[[g_\mu]^0 \otimes S]^1 = g_\mu D^{-1} S^1 = \frac{D}{\mu(D)} D^{-1} S^1 = \frac{S^1}{\mu(D)}$$

come from 3.38 and 5.5.

5.7. **Lemma.** If $S \in \mathfrak{B}$ the equation

$$(5.8) \quad F = [g_\mu]^0 \otimes S$$

determines a solution of the initial-value problem

$$(5.9) \quad \langle \partial_t^k F^1, 0_- \rangle = 0 \quad (\text{for } 0 \leq k \leq m-1)$$

and

$$(5.10) \quad \mu(\partial)F = S;$$

moreover,

$$(5.11) \quad F \in \mathfrak{B} \quad \text{and} \quad \partial^v F \in \mathfrak{B} \quad \text{for all } v \leq m.$$

PROOF. From 5.8 and 5.6 we see that

$$(1) \quad F^1 = \frac{S^1}{\mu(D)},$$

so that 5.9 is obtained by setting $B=S^1$ in [7, (5.7)] (the property $B \in \mathcal{B}_\omega$ comes from 4.9). It only remains to prove 5.10—5.11. From (1) and [7, (5.8)] (again with $B=S^1$) we also obtain

$$(2) \quad \mu(\partial_t) F^1 = S^1.$$

Next, the equations

$$F^1 = \frac{D^m}{\mu(D)} \frac{S^1}{D^m} = \left(\frac{1}{\mu_m} + g_m D^{-1} \right) \frac{S^1}{D_m}$$

are from (1) and [7, (5.5)]; therefore,

$$(3) \quad F^1 = \frac{D}{D^m} D^{-1} \left(\frac{1}{\mu_m} S^1 + g_m D^{-1} S^1 \right)$$

now set

$$(4) \quad G = \frac{1}{\mu_m} S + [g_m]^0 \otimes S.$$

From (4) and 3.38 we obtain $G^1 = \mu_m^{-1} S^1 + g_m D^{-1} S^1$: substituting into (3), we can use 3.45 to write

$$(5) \quad F^1 = Y_m D^{-1} G^1 = ([Y_m]^0 \otimes G)^1;$$

the last equation is from 3.38. Since $g_m(\cdot) \in \mathcal{F}$ we have $[g_m]^0 \in \mathfrak{B}$ (by 1.27), so that $[g_m]^0 \otimes S$ belongs also to $\mathfrak{B}$ (by 2.20); since $S \in \mathfrak{B}$ it now follows directly from (4) that

$$(6) \quad G \text{ belongs to } \mathfrak{B}.$$

In view of (6), we see that $[Y_m]^0 \otimes G$ belongs to $\mathfrak{B}$ (by 2.20); since $F \in \mathfrak{B}$ (by 5.8 and 2.20), we can therefore apply 3.42 to (5):

$$(7) \quad F = [Y_m]^0 \otimes G \quad (\text{from (5)}).$$

From (7) and 3.46 it now follows that $\partial^m F = G$; since $G \in \mathfrak{B}$ it results from 4.12 that

$$(8) \quad \partial^v F \in \mathfrak{B} \quad \text{and} \quad (\partial^v F)^1 = \partial_t^v F^1 \quad (\text{or } 0 \leq v \leq m).$$

From (8) and 5.2 we see that

$$(9) \quad (\mu(\partial) F)^1 = \mu(\partial_t) F^1 = S^1;$$

the second equation is from (2). From (8) it also follows that $\mu(\partial) F$ belongs to $\mathfrak{B}$; we may therefore use 3.42 to infer from (9) that $\mu(\partial) F = S$.

This establishes 5.10. Conclusion 5.11 is immediate from (8), and 5.9 was verified at the beginning of this proof.

5.12. First main theorem. Suppose that $S \in \mathfrak{B}$. If u is a distribution such that $\mu(\partial)u = S$, then $u \in \mathfrak{B}$,

$$(5.13) \quad \partial^k u \in \mathfrak{B}, \quad \text{and} \quad (\partial^k u)^1 = \partial_t^k u^1 \quad (\text{for } 0 \leq k \leq m);$$

moreover,

$$(5.14) \quad (\partial^k u)^1 = D^k u^1 - \sum_{v=1}^{k-1} \langle \partial_t^v u^1, 0_- \rangle D^{k-v} \quad (\text{for } 1 \leq k \leq m).$$

PROOF. Let F be as in 5.8. Since $\mu(\partial)F=S$ and $\mu(\partial)u=S$ it follows from [4, p. 328] the existence of a polynomial $p(\cdot)$ of degree $\leq m-1$ such that $u=F+[p]^0$; consequently, $u \in \mathfrak{B}$ and

$$\partial^m u = \partial^m F + \partial^m [p]^0 = \partial^m F;$$

the last equation is from the fact that $\partial^m [p]^0 = p^{(m)} = 0$ (since the degree of $p(\cdot)$ is $\leq m-1$). Thus, $\partial^m u = \partial^m F$; but $\partial^m F \in \mathfrak{B}$ (by 5.11), so that $\partial^m u \in \mathfrak{B}$: Conclusion 5.13 is now immediate from 4.12. Conclusion 5.14 comes from 5.13 and [7, 4.1].

5.15. Second main theorem. *Let c_v ($v=0, 1, \dots, m-1$) be given numbers. If $S \in \mathfrak{B}$ there exists a unique distribution u such that*

$$(5.16) \quad \mu(\partial)u = S$$

and

$$(5.17) \quad \langle \partial_t^v u^1, 0_- \rangle = c_v \quad (\text{for } 0 \leq v < m).$$

That solution u belongs to $\mathfrak{B}$, satisfies 5.13, and is determined by the equation

$$(5.18) \quad u = [g_\mu]^0 \otimes S + [g_\mu^\lambda]^0,$$

where $g_\mu^\lambda(\cdot)$ is the function determined by

$$(5.19) \quad g_\mu^\lambda = \frac{1}{\mu(D)} \sum_{n=1}^m \mu_n \sum_{v=0}^{n-1} c_v D^{n-v}.$$

PROOF. First, we verify the existence of a solution of the initial-values problem 5.16—5.17. Let $y(\cdot)$ be the solution of the initial-value problem

$$(1) \quad \sum_{k=0}^m \mu_k y^{(k)}(t) = 0 \quad (\text{for } t \in \mathbb{R})$$

subject to the initial conditions

$$(2) \quad y^{(k)}(0) = c_k \quad (\text{for } 0 \leq k < m).$$

It is well-known that such a function $y(\cdot)$ exists; in fact, $y(\cdot)$ is infinitely differentiable; consequently, (2) and [7, 2.21] give

$$(3) \quad \langle \partial_t^k y, 0_- \rangle = c_k \quad (\text{for } 0 \leq k < m).$$

Since $y(\cdot)$ is infinitely differentiable, (1) implies that

$$(4) \quad \mu(\partial)[y]^0 = 0.$$

Let F be the distribution defined by 5.8; we denote by u the distribution

$$(5) \quad u = F + [y]^0.$$

The equations

$$(6) \quad \mu(\partial)u = \mu(\partial)F + \mu(\partial)[y]^0 = S + 0 = S$$

are from (5), 5.10, and (4). On the other hand, (5) and 3.36 give $u^1 = F^1 + y$: consequently,

$$(7) \quad \langle \partial_t^k u^1, 0_- \rangle = \langle \partial_t^k F^1, 0_- \rangle + \langle \partial_t^k y, 0_- \rangle = 0 + c_k = c_k;$$

the last two equations are from 5.9 and (3). From (6)–(7) it follows that (5) defines a distribution u satisfying 5.16–5.17.

Finally, we verify the uniqueness. If u is a distribution satisfying 5.16–5.17, then (since $S \in \mathfrak{B}$) we can use 5.14 to obtain

$$(\mu(\partial)u)^1 = \mu(G)u^1 - \sum_{k=1}^m \mu_k \sum_{v=0}^{k-1} \langle \partial^v u^1, 0_- \rangle D^{k-v};$$

therefore, it results from 5.16–5.17 that

$$S^1 = \mu(D)u^1 - \sum_{k=1}^m \mu_k \sum_{v=0}^{k-1} c_v D^{k-v};$$

solving for u^1 , we can use 5.19 to write

$$(8) \quad u_1 = \frac{S^1}{\mu(D)} + g_\mu^\lambda = ([g_\mu]^\lambda \otimes S + [g_\mu^\lambda]^\lambda)^1;$$

the second equation is from 5.6 and 3.36. Thus, if u is a distribution satisfying 5.16–5.17, it follows from (8) and 3.42 that u is given by 5.18. From 5.12 it results that $u \in \mathfrak{B}$ and u satisfies 5.13. Since we have already verified the existence of a distribution u satisfying 5.16–5.17, our proof is complete.

§ 6. Particular cases

If $b \geq 0$ we denote by $T_b(\cdot)$ the function defined by

$$T_b(t) = \begin{cases} 0 & \text{for } t < b \\ 1 & \text{for } t \geq b \end{cases}.$$

If $a < 0$ we set

$$T_a(t) = \begin{cases} -1 & \text{for } t < a \\ 0 & \text{for } t \geq a \end{cases}$$

As usual, δ_x is the distribution defined by $\delta_x(\varphi) = \varphi(x)$ (for each $\varphi(\cdot)$ in D). We set

$$(6.1) \quad E = \sum_{k=-\infty}^{\infty} \delta_{2k\pi};$$

this series converges in the topology of D' .

6.2. Lemma. E belongs to $\mathfrak{B}$ and

$$(6.3) \quad E^1 = D \sum_{k=-\infty}^{\infty} T_{2k\pi}.$$

PROOF. Note that $E = A + B$, where

$$A = \sum_{k=-\infty}^{-1} \delta_{2k\pi} \quad \text{and} \quad B = \sum_{k=0}^{\infty} \delta_{2k\pi}.$$

It is easily verified that $A \in (\mathcal{L})$ and $B \in \mathfrak{B}_+$, so that $E \in \mathfrak{B}$. Next, observe that $\delta_x = \partial[T_x]^0$ for any x in $\mathbf{R}$; therefore, 6.1 gives

$$(1) \quad E = \sum_{k=-\infty}^{\infty} \partial[T_{2k\pi}]^0 = \partial[f]^0,$$

where

$$(2) \quad f(\cdot) = \sum_{k=-\infty}^{\infty} T_{2k\pi}(\cdot) \quad (\text{see [8, p. 37]}).$$

Since $E \in \mathfrak{B}$, the equations $E^1 = \partial_t[f]^0 = \partial_t f$ follow from (1), 4.10, and 3.36; but $\partial_t f = Df$ (by [6, 5.8]): therefore, $E^1 = Df$. Conclusion 6.3 now comes from [6, 4.12 (with $g=1$)].

6.4. *First example.* To find a distribution u such that

$$(3) \quad \partial^2 u + u = E.$$

Since $E \in \mathfrak{B}$ it follows from 5.12 that $u \in \mathfrak{B}$ and $(\partial^2 u)^1 = \partial_t^2 u^1$; consequently, (3) gives

$$(4) \quad \partial_t^2 u^1 + u^1 = E^1 = D \sum_{k=-\infty}^{\infty} T_{2k\pi};$$

the second equation is from 6.3. The equation (4) is precisely the one that has been solved in [6, 6.7]: the explicit solution is given in the introduction of the present paper.

6.5. *Second example.* In case $S=0$ and $c_{m-1}=1/\mu_m$ with $c_k=0$ for $0 \leq k \leq m-2$, it follows from 5.18—5.19 that the solution of the problem 5.16—5.17 is given by $u = [g_\mu]^0$, where $g_\mu(\cdot)$ is the function defined by 5.5: it is the Green's function of the problem.

6.6. *Third example.* In case $S=\delta$ and $c_k=0$ for $0 \leq k \leq m-1$, it follows from 5.18 that the solution of the problem 5.16—5.17 is given by

$$u = [g_\mu]^0 \otimes \delta = [g_\mu]_+^0;$$

the second equation is from 2.29; equivalently,

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ g_\mu(t) & \text{for } t > 0; \end{cases}$$

see 1.28 and 1.26.

6.7. *Fourth example.* In case $B \subset \mathfrak{B}_+$ then $B = \mathfrak{B}_+$ (see 1.21); it follows from 2.22 that the equation 5.18 becomes

$$u = [g_\mu]_+^0 * B + [g_\mu^\lambda]^0.$$

Added in proof. The results in this paper have been generalized by H. SHULTZ, An algebra of distributions on an open interval. *Transactions of the American Math. Soc.* **169** (1972), 163—181.

References

- [1] H. BREMERMAN, Distributions, complex variables, and Fourier transforms, *Reading, Mass.*, 1965.
- [2] A. CÉSAR DE FREITAS, Sur les distributions qui interviennent dans le calcul symbolique des électrotechniciens (cas des circuits à constantes concentrées) *Rev. Fac. Ciênc. de Lisboa*, 2^a série A, **3** (1955), 279—310.
- [3] L. CHAMBADAL, J-L OVAERT, Cours de Mathématiques, Notions fondamentales d'algèbre et d'analyse, *Paris*, 1966.
- [4] J. HORVÁTH, Topological vector spaces and distributions, vol. I, *Reading, Mass.* 1966.
- [5] G. KRABBE, Operational calculus, *Berlin*, 1970.
- [6] G. KRABBE, An algebra of generalized functions on an open interval: two-sided operational calculus, *Pac. J. Math.*, **42**, (1972).
- [7] G. KRABBE, Initial-value problems involving generalized functions; two-sided operational calculus, *Revue Roumaine Math.*, **7**, (1974).
- [8] L. SCHWARTZ, Théorie des distributions, *Paris*, 1966.
- [9] K. YOSIDA, Functional analysis, *Berlin* 1965.
- [10] A. H. ZEMANIAN, Distribution theory and transform analysis, *New York*, 1965.
- [11] G. KRABBE, An algebra of generalized functions on an open interval; two-sided operational calculus, *Bull. A. M. S.* **77**, (1971) 78—84.

(Received July 11, 1972; in revised form: March 25, 1974)