

Hosszú's functional equation on the unit interval is not stable

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Abstract. We prove that for every $\varepsilon > 0$ there exists a function $f : (0, 1) \rightarrow \mathbb{R}$ satisfying the inequality

$$|f(x + y - xy) + f(xy) - f(x) - f(y)| \leq \varepsilon \text{ for } x, y \in (0, 1),$$

such that for every solution $h : (0, 1) \rightarrow \mathbb{R}$ of the Hosszú's functional equation

$$\sup\{|f(x) - h(x)| : x \in (0, 1)\} = \infty.$$

The same result holds if we replace $(0, 1)$ by any interval with ends 0 and 1.

The functional equation

$$h(x + y - xy) + h(xy) = h(x) + h(y)$$

is referred to as the Hosszú's equation. It is well known (cf. [1]) that for every function $h : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the Hosszú's equation there exists an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ and a constant $C \in \mathbb{R}$ such that

$$h(x) = A(x) + C.$$

By the unit interval we understand any interval with ends 0 and 1 and by $\mathbb{R}_+$ we mean the interval $[0, \infty)$. Since U is closed for operations $(x, y) \rightarrow xy$, $(x, y) \rightarrow x + y - xy$ one can study the Hosszú's equation on U . K. LAJKÓ proved in [2] that if a function $h : U \rightarrow \mathbb{R}$ satisfies the Hosszú's equation then there exists an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ and a constant $C \in \mathbb{R}$ such that

$$(1) \quad h(x) = A(x) + C \text{ for } x \in (0, 1)$$

(h may be arbitrarily chosen on $U \setminus (0, 1)$).

There arises a natural question concerning the stability of the Hosszú's equation. L. LOSONCZI proved in [3] that the Hosszú's equation on $\mathbb{R}$ is stable in the Hyers–Ulam sense, i.e. he obtained the following result.

Theorem L. *If X is a Banach space and $f : \mathbb{R} \rightarrow X$ satisfies the functional inequality*

$$\|f(x + y - xy) + f(xy) - f(x) - f(y)\| \leq \varepsilon \text{ for } x, y \in \mathbb{R}$$

with an $\varepsilon \geq 0$, then there exists a unique function $h : \mathbb{R} \rightarrow X$ satisfying the Hosszú's equation such that

$$\|f(x) - h(x)\| \leq 20\varepsilon \text{ for } x \in \mathbb{R}.$$

Surprisingly, in the case where f is defined on the unit interval, the answer to the question of stability is negative (cf. [4]). For every $\varepsilon > 0$ we can find a function $f_\varepsilon : U \rightarrow \mathbb{R}$ such that

$$|f_\varepsilon(x + y - xy) + f_\varepsilon(xy) - f_\varepsilon(x) - f_\varepsilon(y)| \leq \varepsilon \text{ for } x, y \in U,$$

but which can not be “approximated” by any solution of the Hosszú's equation on the unit interval.

We need the following technical lemma.

Lemma 1. *Suppose that $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies for a certain $L \in \mathbb{R}_+$ the inequalities*

$$(2) \quad |F(x + y) - F(x)| \leq L \text{ for } x \geq y, \ x, y \in \mathbb{R}_+,$$

$$(3) \quad |F(x)| \leq L \text{ for } x \in [0, 6].$$

Let

$$g(a) := \begin{cases} -\ln a & \text{for } a \in \left(0, \frac{1}{2}\right], \\ -\ln(1 - a) & \text{for } a \in \left[\frac{1}{2}, 1\right), \end{cases}$$

$$f(a) := F(g(a)) \quad \text{for } a \in (0, 1).$$

Then

$$(4) \quad |f(a + b - ab) + f(ab) - f(a) - f(b)| \leq 4L \text{ for } a, b \in (0, 1).$$

PROOF. Let $a, b \in (0, 1)$. We are going to prove that (4) is valid. At first we prove that:

- (i). if $a \geq b \geq \frac{1}{2}$ then $g(a + b - ab) = g(a) + g(b)$,
(ii). if $a \geq b, a \geq \frac{1}{2}$ then $|g(ab) - g(b)| \leq 3$,
(iii). if $a \geq b, \frac{1}{2} \geq b$ then $|g(a + b - ab) - g(a)| \leq 3$.

ad (i). We have

$$g(a + b - ab) = -\ln((1-a)(1-b)) = -\ln(1-a) - \ln(1-b) = g(a) + g(b).$$

ad (ii). a). Suppose that $b \geq \frac{1}{2}$.

If $ab \leq \frac{1}{2}$ then $b \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$, $ab \in [\frac{1}{4}, \frac{1}{2}]$, so

$$|g(ab) - g(b)| \leq |g(ab)| + |g(b)| \leq \ln 4 + \left| \ln \left(1 - \frac{1}{\sqrt{2}} \right) \right| \leq 3.$$

If $ab \geq \frac{1}{2}$ then

$$\begin{aligned} |g(ab) - g(b)| &= |\ln(1-ab) - \ln(1-b)| = \left| \ln \left(\frac{1-b+ab}{1-b} \right) \right| \\ &= \left| \ln \left(1 + b \frac{1-a}{1-b} \right) \right| \leq \ln 2 \leq 3. \end{aligned}$$

b). Suppose that $b \leq \frac{1}{2}$. Then

$$|g(ab) - g(b)| = |\ln(ab) - \ln(b)| = |\ln(a)| \leq \ln 2 \leq 3.$$

ad (iii). Let $x^* := 1 - x$. Then $g(x) = g(x^*)$. Making use of the equality

$$g(a + b - ab) = g((a^*b^*)^*) = g(a^*b^*)$$

and interchanging the role of a and b we obtain (iii) from (ii).

Now we show that

$$(5) \quad |F(x) - F(y)| \leq 2L \text{ for } x, y \in \mathbb{R}_+, |x - y| \leq 3.$$

If $x, y \in [0, 6]$ then by (3) relation (5) is obvious. In the other case we may assume that $x \geq y, x \geq 6$. Then $x - y \leq 3 \leq y$ and due to (2) we have

$$|F(x) - F(y)| = |F(y + (x - y)) - F(y)| \leq L \leq 2L.$$

We are going to prove that (4) holds. Without loss of generality we may assume that $a \geq b$. Suppose that $a \geq b \geq \frac{1}{2}$. Then by (i), (ii), (5) and (2)

$$\begin{aligned} &|f(a + b - ab) + f(ab) - f(a) - f(b)| \\ &= |F(g(a + b - ab)) + F(g(ab)) - F(g(a)) - F(g(b))| \\ &\leq |F(g(a + b - ab)) - F(g(a) + g(b))| \\ &\quad + |F(g(a) + g(b)) - F(g(a))| + |F(g(ab) - F(g(b)))| \leq 4L. \end{aligned}$$

Suppose that $a \geq \frac{1}{2} \geq b$. Then due to (ii), (iii) and (5)

$$\begin{aligned} & |f(a+b-ab) + f(ab) - f(a) - f(b)| \\ &= |F(g(a+b-ab)) + F(g(ab)) - F(g(a)) - F(g(b))| \\ &\leq |F(g(a+b-ab)) - F(g(a))| + |F(g(ab)) - F(g(b))| \leq 4L. \end{aligned}$$

Suppose that $\frac{1}{2} \geq a \geq b$. Then $b^* \geq a^* \geq \frac{1}{2}$ and we have

$$\begin{aligned} & |f(a+b-ab) + f(ab) - f(a) - f(b)| \\ &= |f(a^*b^*) + f(a^*+b^*-a^*b^*) - f(a^*) - f(b^*)| \leq 4L. \quad \square \end{aligned}$$

Now we are able to prove the main theorem.

Theorem 1. *Let U be the unit interval. For every $\varepsilon > 0$ there exists a function $f : U \rightarrow \mathbb{R}$ satisfying the inequality*

$$|f(x+y-xy) + f(xy) - f(x) - f(y)| \leq \varepsilon \text{ for } x, y \in U,$$

such that for every solution $h : U \rightarrow \mathbb{R}$ of the Hosszú's functional equation

$$\sup\{|f(x) - h(x)| : x \in U\} = \infty.$$

Moreover, for every $\varepsilon > 0$ and every $K > 0$ there exists a continuous bounded function $f : U \rightarrow \mathbb{R}$ which satisfies the inequality

$$|f(x+y-xy) + f(xy) - f(x) - f(y)| \leq \varepsilon \text{ for } x, y \in U,$$

but such that for every solution $h : U \rightarrow \mathbb{R}$ of the Hosszú's functional equation

$$\sup\{|f(x) - h(x)| : x \in U\} \geq K\varepsilon.$$

PROOF. Without loss of generality we may assume that $\varepsilon = 1$. To prove the first part of the theorem let

$$F(x) := \frac{1}{8} \ln(1+x) \text{ for } x \in \mathbb{R}_+.$$

Then F satisfies (2) and (3) with $L = \frac{1}{4}$. Let

$$f(x) := \begin{cases} F(g(a)) & \text{for } a \in (0, 1), \\ 0 & \text{for } a \in U \setminus (0, 1), \end{cases}$$

where g is the function defined in Lemma 1. We show that

$$|f(a+b-ab) + f(ab) - f(a) - f(b)| \leq 1 \text{ for } a, b \in U.$$

If $a = 0$ or $b = 0$ then this is obvious. For $a, b \in (0, 1)$ the relation holds by Lemma 1. As it is well known (cf. [1], p. 277), an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ is either continuous or has a dense graph in $\mathbb{R} \times \mathbb{R}$. Since f is continuous on $(0, 1)$ and

$$\lim_{a \rightarrow 0} f(a) = \lim_{a \rightarrow 1} f(a) = +\infty,$$

this implies that

$$(6) \quad \sup\{|f(a) - A(a) - C| : a \in (0, 1)\} = \infty.$$

for every additive function A and every constant C . This means that for every solution $h : U \rightarrow \mathbb{R}$ of the Hosszú's functional equation

$$\sup\{|f(a) - h(a)| : a \in U\} = \infty.$$

Now we prove the second part of the theorem. Let

$$F_n(x) := \begin{cases} \frac{1}{8} \ln(1+x) & \text{for } x \in [0, n], \\ \frac{1}{8} \ln(1+n) & \text{for } x \in (n, \infty). \end{cases}$$

and let

$$f_n(a) := \begin{cases} F_n(g(a)) & \text{for } a \in (0, 1), \\ F_n(n) & \text{for } a \in U \setminus (0, 1). \end{cases}$$

One can easily notice that f_n is continuous and by Lemma 1

$$|f_n(a+b-ab) + f_n(ab) - f_n(a) - f_n(b)| \leq 1 \text{ for } a, b \in U.$$

We claim that for every $K > 0$ there is an $n \in \mathbb{N}$ such that

$$\sup\{|f_n(a) - h(a)| : a \in U\} \geq K$$

for all solutions h of Hosszú's equation.

Otherwise, for all $n \in \mathbb{N}$ there were solutions h_n of Hosszú's equation such that

$$(7) \quad \sup\{|f_n(a) - h_n(a)| : a \in U\} < K.$$

Since h_n satisfies the Hosszú's equation, by (1) it has the form

$$(8) \quad h_n(a) = A_n(a) + C_n \text{ for } a \in (0, 1),$$

where A_n is an additive function and C_n is a constant. As f_n is continuous, by (7) and (8) we obtain that A_n is bounded in the interval $[\frac{1}{3}, \frac{2}{3}]$, so there exists $L_n \in \mathbb{R}$ such that $A_n(a) = L_n a$. One can easily check that

$\{f_n(\frac{1}{3})\}$ and $\{f_n(\frac{2}{3})\}$ are bounded sequences. This and (7) implies that $\{C_n\}$, $\{L_n\}$ are bounded sequences. Hence there exist $C, L \in \mathbb{R}$ and an increasing sequence $\{k_n\} \subset \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} C_{k_n} = C \text{ and } \lim_{n \rightarrow \infty} L_{k_n} = L.$$

Since

$$\lim_{n \rightarrow \infty} f_n(a) = f(a) \text{ for } a \in (0, 1),$$

we obtain that

$$|f(a) - La - C| \leq K \text{ for } a \in (0, 1),$$

which contradicts (6). □

References

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