

A growth aspect of entire functions of infinite order

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1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ . We set

$$M(r) = \max_{|z|=r} |f(z)|, \mu(r, f) = \max_{n \geq 0} \{|a_n| r^n\} \quad \text{and} \quad \nu(r, f) = \max_{n \geq 0} \{n | \mu(r, f) = |a_n| r^n\}.$$

As usual, we call $M(r, f)$, $\mu(r, f)$ and $\nu(r, f)$ the maximum modulus, maximum term and the rank of the maximum term respectively for entire function $f(z)$. R_n is known as "*Sprungstellen*" of $\nu(r, f)$ so that $R_n < R_{n+1}$, $\nu(r) = n$ for $R_n \leq r < R_{n+1}$. Let $M(r, f^{(s)})$, $\mu(r, f^{(s)})$ and $\nu(r, f^{(s)})$ correspond similarly to s -th derivative of $f(z)$ i.e. $f^{(s)}(z)$, where

$$f^{(s)}(z) = \sum_{n=s}^{\infty} n(n-1)\dots(n-s+1) a_n z^{n-s}.$$

2. It is known that

$$(2.1) \quad \liminf_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)} = 0 \quad (\rho = \infty). \quad ([4])$$

A result better than (2.1) viz.,

$$(2.2) \quad \liminf_{r \rightarrow \infty} \frac{\log M(r)}{\nu(r)} = 0,$$

for every entire function of infinite order has been proved by SHAH [5]. SHAH and KHANNA [6] proved that for an entire function of infinite order

$$(2.3) \quad \liminf_{r \rightarrow \infty} \frac{\log \{r M(r, f^{(1)})\}}{\nu(r, f)} = 0$$

— a result better than (2.2), since [8]

$$r M(r, f^{(1)}) > M(r) \frac{\log M(r)}{\log r}, \quad r > r_0 = r_0(f).$$

CLUNIE [2] has gone still further to prove that if s is any function of v such that $s(v) = o\left(\frac{v}{\log v}\right)$, then

$$(2.4) \quad \liminf_{r \rightarrow \infty} \frac{\log \{r^s M(r, f^{(s)})\}}{v(r, f)} = 0.$$

The object of this paper is to prove a theorem which is more precise than (2.4) and includes a result of AHMAD [1] as a special case ($s=0$).

3. In what follows, we shall prove the following theorem.

Theorem. For an entire function of infinite order,

$$\lim_{r \rightarrow \infty} \frac{\log \left[\left(r + \frac{\lambda r \log \mu(r)}{v^2(r) J(r)} \right)^s M \left(r + \frac{\lambda r \log \mu(r)}{v^2(r) J(r)}; f^{(s)} \right) \right]}{v(r)} = 0$$

where

- (i) s is a function of v such that $s(v) = o(v/\log v)$.
- (ii) $J(r)$ is any positive function such that

$$\sum_{m=1}^{\infty} \frac{1}{v(R_m) J(R_m)} < \infty$$

and $J(r) \rightarrow \infty$ as $r \rightarrow \infty$, λ being any fixed positive number.

PROOF. We have

$$r^s M(r, f^{(s)}) \leq \sum_{n=s}^{\infty} n(n-1) \dots (n-s+1) |a_n| r^n,$$

in the notations of VALIRON [7; pp. 30—31] for $n \geq p$ $n(n-1) \dots (n-s+1) |a_n| r^n \leq n(n-1) \dots (n-s+1) \mu(r) (r/R_p)^{n-p+1}$.

Therefore, we have

$$\begin{aligned} r^s M(r, f^{(s)}) &\leq \sum_{n=s}^{p-1} n(n-1) \dots (n-s+1) \mu(r) + \\ &+ \sum_{n=p}^{\infty} n(n-1) \dots (n-s+1) \mu(r) (r/R_p)^{n-p+1}, \\ &\leq \mu(r) p^{s+1} + \mu(r) p^{2s} \left[\frac{r}{R_p - r} + \frac{r^2}{(R_p - r)^2} + \dots + \frac{r^{s+1}}{(R_p - r)^{s+1}} \right]. \end{aligned}$$

Take now

$$p = v \left(r + \frac{1}{rv^2(r)} \right) + 1,$$

so that

$$R_p - 1 \leq \frac{1}{rv^2(r)}.$$

We have

$$\begin{aligned} r^s M(r, f^{(s)}) &\leq \mu(r) p^{s+1} + \mu(r) p^{2s} [r^2 v^2(r) + \dots + \{rv(r)\}^{2s+2}] \\ &< \mu(r) \left\{ v \left(r + \frac{1}{rv^2(r)} \right) rv(r) \right\}^{2s+2}, \\ (3.1) \quad \log \{r^s M(r, f^{(s)})\} &< (1+o(1)) \log \mu(r) + (2s+2) \left\{ \log v \left(r + \frac{1}{rv^2(r)} \right) \right\}. \end{aligned}$$

Further, $v(r)$ is constant in the interval $R_n \leq r < R_{n+1}$ ($n=1, 2, 3, \dots$), R_n tends to infinity with n or r [8, p. 30]. Then by (2.1) and our hypothesis $\varrho = \infty$, there is a subsequence of positive integers

$$(3.2) \quad \mathbf{N}: N_1 < N_2 < \dots < N_j < \dots, N_j \rightarrow \infty \quad (j \rightarrow \infty),$$

such that, for given an arbitrary small $\varepsilon > 0$ we have

$$(3.3) \quad \log \mu(R_{N_j}) / v(R_{N_j}) < \varepsilon.$$

There are two possibilities or cases.

Case A. There is an infinite subsequence of $\mathbf{N}$, say

$$(3.4) \quad N_{j_1} < N_{j_2} < N_{j_3} < \dots < N_{j_k} \rightarrow \infty \quad (j_k \rightarrow \infty),$$

which we call (for convenience)

$$(3.5) \quad \mathbf{M}: M_1 < M_2 < \dots < M_k < \dots, M_k \rightarrow \infty \quad (k \rightarrow \infty),$$

and which satisfies the following condition

$$R_{M_k} + 1 > R_{M_k} + \lambda' R_{M_k} \log \mu(R_{M_k}) / v^2(R_{M_k}) J(R_{M_k}), \quad (\lambda' > \lambda),$$

in which case

$$v(R_{M_k} + \lambda' R_{M_k} \log \mu(R_{M_k}) / v^2(R_{M_k}) J(R_{M_k})) = v(R_{M_k}).$$

Case B. There is no infinite subsequence of $\mathbf{N}$ such as $\mathbf{M}$, i.e. for all large j say $j \geq j_0$

$$R_{N_{j+1}} \leq R_{N_j} \left(1 + \frac{\lambda' \log \mu(R_{N_j})}{v^2(R_{N_j}) J(R_{N_j})} \right),$$

in which case either $R_{N_{j+1}} = R_{N_j}$ and then $N_{j+1} \in N$ or $R_{N_{j+1}} > R_{N_j}$.

Our proof consists in showing that, in case A, Theorem is established, while, in case B, there is a contradiction which automatically rules out this case.

Case A. We take the sequence $\mathbf{M} \equiv \{M_k\}$ of (3.5) and use (3.1),

$$\begin{aligned} &\log \{(R_{M_k} + \lambda R_{M_k} \beta)^s M(R_{M_k} + \lambda R_{M_k} \beta; f^{(s)})\} < \\ &< (1+o(1)) \left\{ \log \mu(R_{M_k}) + \int_{R_{M_k}}^{R_{M_k} + \lambda R_{M_k} \beta} (v(x)/x) dx \right\} + \{(2s+2) \log v(R_{M_k} + \lambda' R_{M_k} \beta)\}, \end{aligned}$$

where

$$\beta = \frac{\log \mu(R_{M_k})}{v^2(R_{M_k}) J(R_{M_k})}.$$

Since

$$s = o\left(\frac{v(R_{M_k})}{\log v(R_{M_k})}\right) \quad (k \rightarrow \infty)$$

therefore,

$$\frac{\log \{(R_{M_k} + \lambda R_{M_k} \beta)^s M(R_{M_k} + \lambda R_{M_k} \beta; f^{(s)})\}}{v(R_{M_k})} < (1 + o(1)) \left[\frac{\log \mu(R_{M_k})}{v(R_{M_k})} + \frac{\lambda \log \mu(R_{M_k})}{v^2(R_{M_k}) J(R_{M_k})} \right],$$

i.e.

$$\lim_{k \rightarrow \infty} \frac{\log \left\{ \left(R_{M_k} + \frac{\lambda R_{M_k} \log \mu(R_{M_k})}{v^2(R_{M_k}) J(R_{M_k})} \right)^s M \left(R_{M_k} + \frac{\lambda R_{M_k} \log \mu(R_{M_k})}{v^2(R_{M_k}) J(R_{M_k})} ; f^{(s)} \right) \right\}}{v(R_{M_k})} = 0.$$

This leads to the desired conclusion as explained at the outset of the proof.

Case B. The proof depends on the inference that now the subsequence of integers $N \equiv \{N_j\}$ defined in (3.2), beginning with (say) a certain $M_0 = N_q \equiv N_{j_0}$, consists of all integers without exception or that $\{N_j\}$ for $j \geq q > j_0$, consists of $M_0 = N_q, M_0 + 1, M_0 + 2 \dots M_0 + k$. It is known that

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^r \frac{v(x)}{x} dx.$$

Hence

$$\begin{aligned} \frac{\log \mu(R_{M_0+1})}{v(R_{M_0+1})} &\leq \frac{1}{M_0+1} \left\{ \log \mu(R_{M_0}) + M_0 \log \left(1 + \frac{\lambda' \log \mu(R_{M_0})}{v^2(R_{M_0}) J(R_{M_0})} \right) \right\} < \\ &< \frac{1}{M_0+1} \left\{ \log \mu(R_{M_0}) + \frac{\lambda' \log \mu(R_{M_0})}{M_0 J(R_{M_0})} \right\} < \varepsilon, \end{aligned}$$

and so $M_0 + 1 \in N_j$ ($j = 1, 2, 3 \dots$).

Similarly,

$$M_0 + 2, M_0 + 3, M_0 + 4, \dots \in N_j.$$

Let $M_0 \in N_j$ and $M_0 > N_{j_0}$. Then

$$\begin{aligned} R_{M_0+j} &\leq R_{M_0} \prod_{n=M_0}^{M_0+j-1} \left(1 + \frac{\lambda' \log \mu(R_n)}{v^2(R_n) J(R_n)} \right) \\ &\leq R_{M_0} \left(1 + \frac{\varepsilon \lambda'}{v(R_n) J(R_n)} \right) < a \text{ constant,} \end{aligned}$$

which leads to a contradiction since R_{M_0+j} tends to infinity with j . Hence the alternative *B* is not possible. Thus the theorem is proved.

Remarks. 1. It is an adaptation of Shah's argument [5] combined with (3.1) that forms the basis of the proof of the theorem.

2. It is known [2] that for any entire function of finite or infinite order

$$v(r, f) \equiv v(r, f^{(1)}) \leq \dots \leq v(r, f^{(s)}).$$

Hence it follows that $v(r, f)$ can be replaced by $v(r, f^{(s)})$ where s may have any integer value.

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