

Problems and results on consecutive integers

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To the memory of my friend A. Kertész

In this short survey article I will discuss some questions which occupied me on and off for forty years, some of the problems are classical other special problems. I will also prove some new results.

Denote by $f(k)$ the smallest integer so that the product of $f(k)$ consecutive integers greater than k always contain a prime greater than k . The well known theorem of Sylvester and Schur states $f(k) \leq k$ and I proved $f(k) < \frac{3k}{\log k}$ [4]. Very much stronger results have recently been proved by JUTILA, RAMACHANDRA and SHOREY [15], they showed (improving previous results of TIJDEMAN)

$$(1) \quad f(k) < \frac{c_1 k \log \log \log k}{\log k \log \log k}.$$

(1) is certainly very far from the "truth". It seems sure that $f(k) = o(k^\epsilon)$ and probably $f(k) < c_1 (\log k)^{c_2}$ (the c 's are absolute constants not necessarily the same if they have the same index). These conjectures are inaccessible at present and I have nothing to contribute towards their solution.

In [4] I prove that amongst k consecutive integers greater than k at least $k/6$ of them have a prime factor greater than k . I sharpen and extend this result, but first of all I introduce some notations. Put

$$A_k(m) = \prod p^\alpha, \quad p^\alpha \parallel m, \quad p \leq k$$

and denote by $f(n, k)$ the number of integers $n+i$, $1 \leq i \leq k$ which have at least one prime factor greater than k ; $g(n, k)$ denotes the number of integers $n+i$, $1 \leq i \leq k$ all whose prime factors are $\leq k$ (i.e. the number of integers $n+i$, $1 \leq i \leq k$ with $n+i = A_k(n+i)$). Clearly $f(n, k) + g(n, k) = k$. $U(n, k)$ denotes the number of integers $m \leq n$ with $P(m) \leq k$ where $P(m)$ is the greatest prime factor of m ; $p(m)$ denotes the smallest prime factor of m . $g(n, k, r)$ denotes the number of integers $n+i$, $1 \leq i \leq r$ with $P(n+i) \leq k$.

Theorem 1. Let $\alpha > 1$, $n > k^\alpha - k$, then

$$f(n, k) > k \left(1 - \frac{1}{\alpha} \right) - \frac{2k}{\log k}.$$

We further outline the proof of

Theorem 2. To every $\alpha > 1$ there is an $\varepsilon_\alpha > 0$ so that for $k > k_0(\alpha)$ and $n > k^\alpha - k$

$$f(n, k) > k \left(1 - \frac{1}{\alpha} + \varepsilon_\alpha \right).$$

These theorems are no doubt very far from being best possible. DE BRUIJN [1] and others proved that there is a c_α so that for $k \rightarrow \infty$

$$(2) \quad U(k^\alpha, k) = (c_\alpha + o(1))k^\alpha,$$

and in fact as $\alpha \rightarrow \infty$ c_α tends to 0 a little faster than $(([\alpha] + 1)!)^{-1}$. It is well known and easy to see that for $1 \leq \alpha \leq 2$ $c_\alpha = 1 - \log \alpha$ but for $\alpha > 2$ though c_α can be calculated explicitly the formula for it is quite complicated. It seems certain that for $\log n = (\alpha + o(1)) \log k$

$$(3) \quad f(n, k) = (1 - c_\alpha + o(1))k.$$

Ramachandra, Shorey and Tijdeman proved (their paper will appear in the *Journal für reine u. angew. Math.*) that if $n > \exp(c(\log k)^2)$ then $f(n, k) \geq k - \pi(k)$ and Shorey proved (will appear in *Acta Arithmetica*) that if $n > \exp k^\varepsilon$ then $f(n, k) > k \left(1 - \frac{ck \log \log k}{(\log k)^2} \right)$.

It seems even harder to get non trivial upper bounds for $f(n, k)$. We have

Theorem 3. For $n \leq k^\alpha - k$ we have

$$f(n, k) < k(\alpha - 1) + \frac{k}{\log k}.$$

For $\alpha > 2$ this is trivial and I do not know any non trivial upper bound for $f(n, k)$ if $n > k^\alpha$, $\alpha > 2$. I can not even prove that there is an absolute constant $c^{(\alpha)}$ so that for every $n < k^\alpha$

$$(4) \quad g(n, k) > c^{(\alpha)}k, \quad \text{or} \quad f(n, k) < (1 - c^{(\alpha)})k.$$

(4) is related to another old conjecture of mine which often annoyed me greatly: Is it true that to every α there is a $n_0(\alpha)$ so that every $n > n_0(\alpha)$ is the sum of two positive integers $a + b = n$ with $P(a \cdot b) < n^{1/\alpha}$. I have been unable to prove this for $\alpha > 2$. By a heuristic independence argument one would expect that the number of solutions of

$$n = a + b, \quad P(a \cdot b) < n^{1/\alpha}$$

is $(c_\alpha^2 + o(1))n$. Similarly one would expect that the events $P(n) \leq k$ and $P(n+1) \leq k$

are independent. More generally I conjecture that the number of integers $n < k^\alpha$ for which

$$P\left(\prod_{i=1}^r (n+i)\right) \leq k \quad (r, \alpha \text{ fixed}, k \rightarrow \infty)$$

is $(c_\alpha + o(1))k^\alpha$. I have made no progress with this conjecture not even for $r=2$. We obtain from (2) by a simple averaging process

$$\lim_{k \rightarrow \infty} \frac{1}{rk^\alpha} \sum_{n=1}^{k^\alpha} g(n, k, r) = c_\alpha.$$

Independence arguments suggest

$$(5) \quad \lim_{k \rightarrow \infty} \frac{1}{r^2 k^\alpha} \sum_{n=1}^{k^\alpha} g(n, k, r)^2 = c_\alpha^2$$

but I could not prove (5).

Denote by n_k the smallest integer with $f(n_k, k) = k$. By the *Chinese remainder theorem* we immediately obtain $n_k < \prod_{i=0}^{k-1} p_{s+i}$ where $k < p_s < p_{s+1} < \dots$ are the consecutive primes greater than k . It is easy to get a very much better upper bound for n_k . Observe that the number of integers $m < n_k$ with $P(m) \leq k$ is at least n_k/k (since every set of k consecutive integers $\leq n_k$ all whose prime factors are $\leq k$). Thus we obtain from (2) by a simple computation that for $k > k_0$

$$(6) \quad n_k < k^{\log k / \log \log k}.$$

I think (6) is fairly sharp. I feel sure that for every $\varepsilon > 0$ and $k > k_0(\varepsilon)$

$$(7) \quad n_k > \exp((\log k)^{2-\varepsilon}).$$

I am very far from being able to prove (7), in fact can not even show $n_k > k^{2+\varepsilon}$ which seems a ridiculously weak result. The best that I can show is

$$n_k > k^2 \exp((\log k)^c)$$

for a certain $c > 0$.

I once conjectured that for $n \geq 2k$ $\binom{n}{k}$ is always divisible by at least one of the integers $n-i$, $0 \leq i < k$. SCHINZEL disproved this conjecture and our results with SCHINZEL [18] makes it likely that there is a k_0 so that for $k > k_0$ the conjecture only holds if $k = p^\alpha$. I now conjecture that there is a $c < 1$ so that for $n \geq 2k$ $\binom{n}{k}$ always has a divisor d satisfying $cn < d \leq n$.

Several mathematicians investigated $P\left(\binom{n}{k}\right)$ ([5]). I conjecture that for every $\alpha < \frac{1}{2}$ and $k > k_0(\alpha)$

$$(8) \quad P\left(\binom{n}{k}\right) > \min \{ \exp(c_\alpha k^\alpha), \exp c_\alpha (\log n)^\alpha \}.$$

The results of RANKIN and myself [6] imply that for $k < c \log n$

$$\liminf P\left(\binom{n}{k}\right)/\log n = 0,$$

but perhaps if $\alpha > 0$ then

$$P\left(\binom{n}{k}\right) > n^{c_\alpha}$$

for every $k > (\log n)^\alpha$ and $c_\alpha \rightarrow 1$ as $\alpha \rightarrow 1$.

It is known that $P(n(n+1)) > c \log \log n$ and perhaps

$$(9) \quad \lim_{n \rightarrow \infty} P(n(n+1))/(\log n)^{2-\varepsilon} = \infty$$

for every $\varepsilon > 0$. (9) if true is certainly very deep. On probability grounds I expect (by (2)) that for every $\varepsilon > 0$ and infinitely many n

$$(10) \quad P(n(n+1)) < (\log n)^{2+\varepsilon}.$$

(10) if true is no doubt very deep. I can not even prove that

$$\liminf \log P(n(n+1))/\log n = 0.$$

Let $h(k)$ be the greatest integer so that for every $n \geq h(k)$

$$P\left(\prod_{i=1}^k (n+i)\right) > h(k).$$

It is not difficult to prove $h(k) > ck \log k$ but no doubt $h(k)$ increases much faster and it seems very difficult to get a good estimation for $h(k)$. $h(k) > k^c$ is certainly true for every c if $k > k_0(c)$.

$$h(k)/k \log k \rightarrow \infty$$

follows from (1).

A well known conjecture of Cramer states

$$(11) \quad \limsup \frac{p_{n+1} - p_n}{(\log n)^2} = 1$$

where $p_1 < p_2 < \dots$ is the sequence of consecutive primes. If (11) is true the order of magnitude of $h(k)$ probably will be $\exp((\log k)^2)$. Let in fact p_{n_k} be the smallest prime for which $p_{n_k+1} - p_{n_k} > k$. Clearly

$$h(k) \leq P\left(\prod_{i=1}^k (p_{n_k} + i)\right).$$

The problems on $h(k)$ are of course connected with the conjecture (8). Related problems were investigated by ECKLUND, EGGLETON and SELFRIDGE [2].

For fixed n

$$P\left(\prod_{i=1}^k (n+i)\right) = u_n(k)$$

is clearly a non decreasing function of k . Denote by $p^{(n)}$ the least prime greater than n . Clearly $u_n(k)$ ceases to be interesting for $k \geq p^{(n)} - n$, since it then equals the greatest prime $\leq n+k$. It might be of some interest to determine the maximum number of possible increases of $u_n(k)$, $1 \leq k < p^{(n)} - n$ as a function of n — can it be as big as $\log \log n$? or $\log n$? The answer to the first question is probably yes to the second one no.

$$P\left(\prod_{j < p_{i+1} - p_i} (p_i + j)\right) = P_i$$

also is an interesting function. I do not think that P_i tends to infinity, in fact perhaps $P_i = 3$ for infinitely many i in other words there are infinitely many integers $n = 2^\alpha 3^\beta$ for which $n+1$ and $n-1$ are both primes. On the other hand for most values of i P_i tends to infinity probably quite fast — the explanation of my vagueness is ignorance.

By a simple averaging process I deduced [8]

$$(12) \quad \min_{1 \leq i \leq k} A_k(n+i) < ck.$$

I conjectured that (12) holds for every $c > 0$ if $k > k_0(c)$. This conjecture always seemed very interesting to me, but unfortunately I was unable to make any progress with it. More generally it would be interesting to investigate

$$A(k) = \max_n \min_{1 \leq i \leq k} A_k(n+i).$$

$A(k) \rightarrow \infty$ as $k \rightarrow \infty$ is not hard to prove but I have no idea how fast $A(k)$ tends to infinity. A stronger conjecture than (12) is

$$\sum_{i=1}^k \frac{1}{A_k(n+i)} \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

perhaps even

$$\sum_{i=1}^k \frac{1}{A_k(n+i)} \geq (1+o(1)) \log k.$$

It is easy to deduce by Turán's method [19] (using second moments) that the normal order of

$$\sum_{i=1}^k \frac{1}{A_k(n+i)} \quad (\text{as } n \text{ varies}) \text{ is } (1+o(1))e^{-c} \frac{\pi^2}{6} \frac{k}{\log k}$$

where c is Euler's constant. Clearly

$$\min_n \max_{1 \leq i \leq k} A_k(n+i) = k.$$

I do not know how large is the normal order of this function. Probably it is more than k^c for every c if $k \gg k_0(c)$. A well known result of MAHLER [17] implies that for every k and $\varepsilon > 0$

$$(13) \quad A_k \left(\prod_{i=1}^k (n+i) \right) < n^{1+\varepsilon}.$$

On the other hand it is easy to see that for infinitely many n

$$(14) \quad A_3(n(n+1)) > cn \log n.$$

It would be very interesting to sharpen (13) and (14) but I could not even prove that for some fixed k

$$\limsup_{n \rightarrow \infty} A_k \left(\prod_{i=1}^k (n+i) \right) / n \log n = \infty.$$

By an averaging process it is not difficult to prove that

$$\lim_{k \rightarrow \infty} A_k \left(\binom{n}{k} \right)^{1/k} = c$$

holds for almost all n and large k .

PLEASANT proved that $v \left(\binom{n}{k} \right) \geq v(n)$ always holds (unpublished) ($v(n)$ denotes the number of distinct prime factors of n). SELFRIDGE and I conjectured [7]

$$\max_{1 \leq k \leq n} v \left(\binom{n}{k} \right) / k \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

but we have not even proved

$$(15) \quad \max_{1 \leq k \leq n} \left(v \left(\binom{n}{k} \right) - k \right) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

(15) follows from recent results of SHOREY and RAMACHANDRA (will appear in *Acta Arithmetica*).

Let $\alpha < 1$, $k = n^{\alpha + o(1)}$. Very likely

$$(16) \quad v \left(\binom{n}{k} \right) = (1 + o(1))k \sum_{p \leq n} \frac{1}{p} = (1 + o(1))k \log \frac{1}{\alpha}.$$

(16) if true is certainly deep.

Put $t_1 = (\log n)^{1-\varepsilon}$, $t_2 = (\log n)^{1+\varepsilon}$. It is easy to see from the Chinese remainder theorem that

$$\limsup_{n \rightarrow \infty} \frac{1}{t_1 \log \log n} \sum_{i=1}^{t_1} v(n+i) = \infty.$$

Very likely

$$(17) \quad \lim_{n \rightarrow \infty} \frac{1}{t_2 \log \log n} \sum_{i=1}^{t_2} v(n+i) = 1.$$

I have no idea how to attack (17), perhaps (17) already holds for $t_2 = \log n$, for some related questions see [7].

I proved [9] that

$$\max_{1 \leq k \leq \frac{n}{2}} v \left(\binom{n}{k} \right) = (1 + o(1)) \frac{n \log 2}{\log n}$$

and the maximum is assumed for $k = (1 + o(1)) \frac{n}{2}$. As k increases from 1 to $\frac{n}{2}$, $v\left(\binom{n}{k}\right)$ has of course a tendency to increase, but no doubt for every c and $n > n_0(c)$ there are values of k $\left(1 \leq k < \frac{n}{2}\right)$ for which

$$v\left(\binom{n}{k}\right) > v\left(\binom{n}{k+1}\right) - c.$$

GRIMM [14] conjectured that if $n+1, \dots, n+t$ are consecutive composite numbers then for $1 \leq i \leq t$ there is a $p_i | n+i$, $p_i \neq p_j$ for $i \neq j$. SELFRIDGE and I [10] proved this for $t = o(\log n)$, we also showed that Grimm's conjecture if true is certainly very deep. Ramachandra, Shorey and Tijdeman proved that Grimm's conjecture holds for $t = [c(\log n / \log \log n)^3]$. Their paper will soon appear in *J. reine u. angew. Math.* For further problems and results in this direction I have to refer to the original papers which are quoted in the paper of Ramachandra, Tijdeman and Shorey.

Selfridge and I proved that the product of consecutive integers is never a power. Our proof will soon appear in the *Illinois Journal of Mathematics*. The following problem is of interest here: Denote by $H_k(n, l)$ the smallest integer i_k for which

$$(n+i_1) \dots (n+i_k) = x^l, \quad 0 \leq i_1 < \dots < i_k = H_k(n, l)$$

is solvable. Our theorem with Selfridge asserts $H_k(n, l) \geq k$ for every $k > 1$, $n \geq 1$ and $l > 1$.

There seems little doubt that this can be improved a great deal, but as far as I know this question has not yet been seriously investigated.

Selfridge and I in fact prove that for every k and $l > 1$ $\prod_{i=1}^k (n+i)$ has a prime factor $p \equiv k$ with $p^x \parallel \prod_{i=1}^k (n+i)$, $\alpha \not\equiv 0 \pmod{l}$. We conjecture that in fact for $k > 2$ there is such a prime with $\alpha = 1$, we made no progress with this deep conjecture. I once thought that for every $k > 1$ $\prod_{i=1}^k (n+i)$ has a prime p for which $p \parallel \prod_{i=1}^k (n+i)$, but Mahler pointed it out to me that since $x^2 - 8y^2 = 1$ has infinitely many solutions there are infinitely many values of n for which every prime factor of $n(n+1)$ occurs with an exponent greater than one. I would guess though that the number of such integers $n < x$ is less than $(\log x)^c$. Probably there always is a p with $p^x \parallel n(n+1)$, $\alpha \leq 2$, in fact if $a_1 < a_2 < \dots$ is the sequence of integers all whose prime factors are $\equiv 3$ then one would conjecture $\lim_{i \rightarrow \infty} (a_{i+1} - a_i) = \infty$, and in fact $a_{i+1} - a_i > c_1 i^{c_2}$.

Put

$$S_r(m) = \prod p^x, \quad p^x \parallel m, \quad \alpha > r.$$

Let k be fixed. Determine or estimate

$$\limsup_{n \rightarrow \infty} \log S_1 \left(\prod_{i=1}^k (n+i) \right) / \log n = \alpha_k.$$

Trivially $\alpha_k \leq k$ and by Mahler's remark $\alpha_k \geq 2$. It would be interesting to improve these bounds. It is fairly sure that for $k > 2$, $\alpha_k < k$ and I would not be surprised if $\alpha_k = 2$ for every k .

Observe that

$$\frac{m}{S_1(m)} = \prod_{p \parallel m} p.$$

RIGGE [13] and a few month later I proved in 1939 that the product of consecutive integers is never a square. Our proof was based on the fact that for $k > 4$ the k integers $A_k(n+i)/S_1(A_k(n+i))$, $i=1, \dots, k$, can not all be different. Two strengthenings of this result: Is it true that for $r > 1$ there is a $k_0(r)$ so that for $k > k_0(r)$ the k integers $A_k(n+i)/S_r(A_k(n+i))$ can not all be different? I have not decided this even for $r=2$. Is there a $c < 1$ so that for $k > k_0(c)$ the ck integers $A_k(n+i)/S_1(A_k(n+i))$, $1 \leq i \leq ck$, can not all be different?

Let $h(n)$ be the largest integer for which the integers $\frac{n+i}{S_1(n+i)}$, $i=1, \dots, h(n)$ are all distinct. It is very likely that infinitely often $h(n) > cn^{\frac{1}{3}-\epsilon}$ but I do not see at present whether $h(n)/n^{1/2} \rightarrow 0$ is true or false.

Denote by $G(k)$ the largest integer for which there are $G(k)$ consecutive integers $n+i$, $1 \leq i \leq G(k)$ for which the integers $A_k(n+i)$, $1 \leq i \leq G(k)$ are all different. BASIL GORDON and I proved some time ago the following

Theorem 4. *Let $2=p_1 < p_2 < \dots < p_s \leq k < p_{s+1} < p_{s+2}$ be the sequence of consecutive primes. Then*

$$p_{s+2} - 2 \leq G(k) \leq (2 + o(1))k.$$

We have no counterexample to $G(k) = p_{s+2} - 2$. Since our proof has never been published I give it here in full detail.

Before we prove our Theorems we state a few disconnected problems on consecutive integers. Selfridge and I conjectured that for $n \geq 2k$ $\binom{n}{k}$ has a prime factor $< \frac{n}{2}$ the sole exception is $\binom{7}{3}$. Ecklund proved this conjecture [3]. Selfridge and I proved that there is an absolute constant c so that for $n \geq 2k$ and $k > k_0(c)$ $\binom{n}{k}$ has a prime factor $< \frac{n}{k^c}$ [10]. Selfridge and I conjectured that perhaps c can be taken as 1. Very likely this is best possible since we almost certainly have

$$(18) \quad \limsup_{n \rightarrow \infty} p \left(\binom{n}{k} \right) / n = \frac{1}{k}.$$

(18) if true will certainly be very hard to prove. Ecklund, Selfridge and I investigated the smallest integer $m_k \geq k+1$ for which all prime factors of $\binom{m_k}{k}$ are greater than k [12], our lower bounds for m_k are poor, we only get $m_k > k^{1+c}$.

In a forthcoming paper in *Mathematics of Computation* GRAHAM, RUZSA, STRAUS and I prove that to every two odd primes p and q there are infinitely many values of n for which $p \nmid \binom{2n}{n}$ and $q \nmid \binom{2n}{n}$. We could not extend this for three primes.

We also state the following *problem*: Is it true that there is an absolute constant c so that

$$\sum \frac{1}{p} < c \quad \text{where } p \nmid \binom{2n}{n} \quad \text{and } p < 2n?$$

Many rather special problems can be stated on binomial coefficients — here are a few which occurred me recently. Denote by α_r , $r=0, 1, \dots$ the density of integers n for which there are exactly r squarefree integers in the sequence $\binom{n}{k}$, $1 \leq k \leq n-1$.

It is not difficult to prove that α_r exists and $\sum_{r=0}^{\infty} \alpha_r = 1$. The following question seems much more difficult: Denote by $s(n)$ the number of squarefree integers amongs the sequence $\binom{n}{k}$, $1 \leq k \leq n-1$. Probably $s(n)$ can not be very large, perhaps $s(n) = o(n^{\epsilon})$ for every $\epsilon > 0$, but I have not obtained any sharp results.

The prime number theorem implies

$$\max_{1 \leq m \leq n} v(m) = (1 + o(1)) \frac{\log n}{\log \log n}.$$

It is very likely that for every k

$$(19) \quad \max_{1 \leq m \leq n} v \left(\prod_{i=1}^k (m+i) \right) = (1 + o(1)) \log n / \log \log n.$$

(19) if true will be very hard to prove. Put ($d(n)$ is the number of divisors of n)

$$D(n) = \max_{1 \leq m \leq n} d(m).$$

$D(n)$ has been studied among others by RAMANUJAN in his paper "On highly composite numbers" (see Collected papers of S. RAMANUJAN, Cambridge, 1927). Almost certainly

$$\limsup_{n \rightarrow \infty} \max_{1 \leq m \leq n} d(m(m+1)) / D(n) = \infty$$

but probably for a certain $c > 0$

$$\max_{1 \leq m \leq n} d(m(m+1)) < n^c D(n).$$

All these questions seem very difficult. On the other hand it is a simple exercise to prove that for every k ($\sigma(n)$ is the sum of the divisors of n)

$$\max_{1 \leq m \leq n} \frac{\sigma \left(\prod_{i=1}^k (m+i) \right)}{\prod_{i=1}^k (m+i)} - \max_{1 \leq m \leq n} \frac{\sigma(m)}{m} \rightarrow 0.$$

Put $A_k = \prod_{i=1}^k p_i$, where p_i are the consecutive primes. The prime number theorem implies $A_k^{1/p_k} \rightarrow e$. Denote by $g(k)$ the number of integers $m < A_k$ for which

$(m+i, A_k) > 1$ for $i=1, 2, \dots, p_k$. $g(k)$ certainly tends to infinity with k but perhaps

$$\lim_{k \rightarrow \infty} g(k)^{1/p_k} < e?$$

Many similar questions can be asked which are connected with the growth of $p_{i+1} - p_i$. Selfridge and I asked: Is it true that to every r there is a $k_0(r)$ so that for $k > k_0(r)$ there is an $m < A_k$ with $v((m+i, A_k)) > r$ for $i=1, 2, \dots, p_k$? This problem seems to be surprisingly difficult and is perhaps not affirmative for every r . A related question states as follows:

Denote by $K(n)$ the largest integer so that for every $1 \leq i \leq K(n)$ $v(n+i) > \log \log n$. The Chinese remainder theorem implies $K(n) \geq (1+o(1)) \log n / (\log \log n)^2$ and this is the best lower bound that I can get. I have no non trivial upper bound for $K(n)$. On probability grounds one would expect

$$\limsup_{n \rightarrow \infty} K(n) / \log n = \frac{1}{\log 2}.$$

An old conjecture of Catalan states that 8 and 9 are the only consecutive powers. This problem seemed intractable. A few months ago TIJDEMAN proved that there is a computable constant c so that two consecutive integers greater than c can not be both powers and CHOUDNOVSKI proved that there is an explicitly given function $L(n)$ so that if $1 = a_1 < a_2 < \dots$ is the sequence of powers then $a_{n+1} - a_n > L(n)$ holds for every n .

Now we prove our Theorems. Clearly $\binom{n}{k} > \frac{n^k}{k^k}$, also a well known lemma asserts that $p^\beta \parallel \binom{n}{k}$ implies $p^\beta \leq n$. Now put $\binom{n}{k} = u_1 \cdot u_2$ where $P(u_1) \leq k$ and $p(u_2) > k$. From $\pi(k) < \frac{2k}{\log k}$ we obtain $u_1 < n^{2k/\log k}$. Thus

$$(20) \quad n^{f(n-k, k)} > u_2 > n^{k - \frac{2k}{\log k}} \cdot k^{-k}.$$

(20) immediately implies Theorem 1 by $n > k^x$.

It might be of some interest to investigate

$$\min_{p^x \parallel \binom{n}{k}} p^x = V(n, k) \quad \text{and} \quad \min_{\substack{p^x \parallel \binom{n}{k} \\ p \leq k}} p^x$$

and try to obtain upper and lower bounds for these functions. I had no time to investigate whether one can get any non trivial results.

We now outline the proof of Theorem 2. Let $n-k+1 \leq m_1 < \dots < m_{f(n-k, k)} \leq n$ be the integers which have a prime factor greater than k . In the proof of Theorem 1 we crudely assumed that the m_i are entirely composed of the primes greater than k and obtained Theorem 1 from (20). In fact we can improve this estimate by the following

Lemma 1. *To every $\eta_1 > 0$ there is an $\eta_2 > 0$ so that for $k > k_0(\eta_1, \eta_2)$ all but $\eta_1 k$ of the integers $n-k+1 \leq m \leq n$ have a prime factor p satisfying $k^{\eta_2} < p \leq k$.*

The proof of Lemma 1 follows easily by the *sieve of Eratosthenes* and will be left to the reader.

From Lemma 1 we obtain that at least $f(n-k, k) - \eta_1 k$ of the integers m_i have a prime factor p satisfying $k^{\eta_2} < p \leq k$. Thus (20) can clearly be replaced by the following sharper inequality

$$(21) \quad n^{f(n-k, k)} > n^{k - \frac{2k}{\log k}} \cdot k^{-k} \cdot k^{\eta_2(f(n-k, k) - \eta_1 k)}.$$

Using $n \geq k^x$ (21) easily implies Theorem 2.

Theorem 3 is nearly trivial. We evidently have (since $\binom{n}{k}$ is an integer)

$$(22) \quad \prod_{i=0}^{k-1} (n-i) \cong k!$$

where the dash indicates that we remove from $n-i$ all prime factors greater than k . From (22) and $n \leq k^x$ we have

$$k^{kx} \cong n^k > k! k^{f(n-k, k)} > k^{k+f(n-k, k)} e^{-k}$$

which proves Theorem 3.

The most difficult proof is our proof with GORDON of Theorem 4. First we observe $G(k) \cong p_{s+2} - 2$. Clearly the $p_{s+2} - 2$ integers $A_k(2), A_k(3), \dots, A_k(p_{s+2} - 1)$ are all distinct since $A_k(i) = i$ for all of them except for $A_k(p_{s+1})$ which is 1. The proof of the upper bound of Theorem 4 is a little more difficult. Suppose that the $G(k)$ integers $A_k(n+i), 1 \leq i \leq G(k)$ are all different. Observe that

$$(24) \quad \prod' A_k(n+i) \cong G(k)! G(k)^{-\pi(k)} > G(k)^{G(k)} \cdot e^{-G(k)} \cdot G(k)^{-\pi(k)},$$

where in $\prod' A_k(n+i), 1 \leq i \leq k$, we omit for each $p \leq k$ the integer $n+i$ which is divisible by the highest power p^x (if there are several such $(n+i)$'s we omit the greatest one). (24) follows immediately since by assumption the $A_k(n+i), 1 \leq i \leq G(k)$ are all distinct. On the other hand

$$(25) \quad \prod' A_k(n+i) \cong G(k)! \left(\prod_{k < p \leq G(k)} p \right)^{-1} < G(k)^{G(k)} \cdot e^{-2G(k) + k + o(G(k))}.$$

The first inequality of (25) follows from the fact that by Legendre's formula all primes $p \leq k$ occur in a higher power in $G(k)!$ than in $\prod' A_k(n+i)$ and by definition $A_k(n+i)$ has no prime factor greater than k ; the second inequality of (25) follows from Stirling's formula. (24) and (25) implies

$$e^{-G(k) + k + o(G(k))} > G(k)^{-\pi(k)}$$

which by the prime number theorem implies $G(k) \leq (2 + o(1))k$ and hence the proof of Theorem 4 is complete. It seems very likely that the upper bound in Theorem 4 can be improved but we have not been successful in our attempts.

To finish this paper I state one more *problem*: Determine or estimate the smallest integer $H(n)$ so that one can find two subsets of the integers $n+1, n+2, \dots, n+H(n)$ whose product is equal.

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