

On $p^{\omega+n}$ -projective abelian p -groups

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In memoriam Andor Kertész

Among the abelian p -groups without elements of infinite heights, only the direct sums of cyclic p -groups and the torsion-complete p -groups have satisfactory structure theories. In addition, a great deal is known of the direct sums of torsion-complete p -groups (see e.g. [2], section 73) and of the $p^{\omega+1}$ -projective p -groups (see FUCHS and IRWIN [3]); in both cases, the socles—if viewed as vector spaces with valuation—determine the group structures.

The purpose of this note is to pursue the idea developed in [3] and to extend the results of [3] to the class of $p^{\omega+n}$ -projective p -groups.

Recall that a $p^{\omega+n}$ -projective p -group (where ω stands for the first infinite ordinal and n is a positive integer) can be defined in one of the following equivalent ways (cf. NUNKE [5], BENABDALLAH, IRWIN and RAFIQ [1]): it is a p -group A satisfying

- (a) $p^{\omega+n} \text{Ext}(A, G) = 0$ for all groups G ;
- (b) it contains a p^n -bounded subgroup P (i.e. $p^n P = 0$) such that A/P is a direct sum of cyclic groups;
- (c) it is isomorphic to F/U for some direct sum F of cyclic p -groups and a p^n -bounded subgroup U of F .

Our present approach is similar to the one developed in [3]: we shall view the p^n -socle $A[p^n] = \{a \in A \mid p^n a = 0\}$ of A as an abelian group with valuation where the values are given by the height function $h(a)$. The main result states that two $p^{\omega+n}$ -projective p -groups are isomorphic exactly if their p^n -socles are isometric as valued abelian groups (Theorem 5).

We have found for this theorem a more direct proof than the one used in [3] to prove the analogous theorem. In the present case, however, we have been unable to get the same amount of information about the structures of $p^{\omega+n}$ -projectives. As a matter of fact, several questions settled in the case of $p^{\omega+1}$ -projectives have been left open, of which the most interesting is John Irwin's conjecture on the ubiquity of $p^{\omega+n}$ -projectives: does every p -group which is not $p^{\omega+n-1}$ -projective contain a proper $p^{\omega+n}$ -projective p -group (i.e. one which fails to be $p^{\omega+n-1}$ -projective)?

By a group we shall mean throughout an abelian p -group A where p is a prime. For the basic definitions and fundamental results we refer to our book [2]. As usual, $p^\sigma A$ is defined for all ordinals σ by setting $p^{\sigma+1}A = p(p^\sigma A)$ and $p^\sigma A = \bigcap_{\sigma < \rho} p^\rho A$ whenever

ϱ is a limit ordinal. For the sake of simplicity, we assume A reduced, i.e. $p^\tau A = 0$ for some ordinal τ . By the *height* $h(a)$ of an element $a \neq 0$ in A is meant the ordinal σ if $a \in p^\sigma A \setminus p^{\sigma+1} A$. We set $h(0) = \infty$.

A subgroup N of A is said to be *nice* if $p^\sigma(A/N) = (p^\sigma A + N)/N$ for all ordinals σ . I.e. every coset of $A \bmod N$ can be represented by an element $a \in A$ of the same height.

If P is a p^n -bounded subgroup of A and G is an arbitrary subgroup with $G[p^n] = P$, we then say G is *supported* by P . In particular, $p^\sigma A$ is supported by $p^\sigma A[p^n]$.

1. Let A be a p -group. By a *valuation* of A we shall mean a function $v: A \rightarrow \Gamma \cup \{\infty\}$ (where Γ denotes the class of ordinals and the symbol ∞ is regarded as larger than any ordinal) such that

- (i) $v(a) = \infty$ if and only if $a = 0$;
- (ii) $v(ma) = v(a)$ or $> v(a)$ according as the integer m is prime to p or not;
- (iii) $v(a+b) \geq \min(v(a), v(b))$ for all $a, b \in A$.

Two groups with valuations are called *isometric* if there is a value-preserving isomorphism between them. A *morphism* between two valued groups is a group theoretical homomorphism which does not decrease values.

Let C be a subgroup of a group A with valuation v . Then the restriction of v to C makes C into a group with valuation. The quotient A/C carries the induced valuation: $v(a+C) = \sup_{c \in C} v(a+c)$ or $= h(a+C)$ = the height of the coset $a+C$ in the quotient A/C whichever is larger, and the canonical map $A \rightarrow A/C$ is a morphism of valued groups. A reduced p -group A can always be regarded as being equipped with the valuation h = height function, or if it is a subgroup of a reduced p -group G , it can be furnished with the valuation inherited from G .

A cyclic group $\langle a \rangle$ of order p^n is said to be *fundamental* if its valuation is given by

$$v(x) = v(a) + h(x) \quad (x \in \langle a \rangle)$$

where h denotes the height function in $\langle a \rangle$. More generally, a p -group A with valuation v will be called σ -*fundamental* ($\sigma \in \Gamma$) if in the subgroup $A^\sigma = \{a \in A \mid v(a) \geq \sigma\}$ the valuation is given by $\sigma + h(a)$ with h computed in A^σ . (A cyclic p -group $\langle a \rangle$ is thus fundamental exactly if it is $v(a)$ -fundamental.)

Let A_i ($i \in I$) be p -groups with valuations. By their direct sum $A = \bigoplus A_i$ is meant their group theoretical direct sum equipped with the valuation

$$v(\sum a_i) = \min_i v(a_i) \quad (a_i \in A_i).$$

In particular, if all A_i are fundamental cyclic groups $\langle a_i \rangle$ of order p^{n_i} , with $v(a_i) = \alpha_i$, then their direct sum A will be free in the sense that any function f from the basis $\{a_i\}_{i \in I}$ into a p^n -bounded valued group G such that $v(a_i) \leq v(f(a_i))$ extends to a unique morphism $A \rightarrow G$ (as valued groups).

2. Our study starts with the characterization of those p^n -bounded groups P with valuation which can appear as subgroups in $p^{\omega+n}$ -projective p -groups A such that A/P is a direct sum of cyclics.

Theorem 1. For a p^n -bounded group P with valuation to be a subgroup of a $p^{\omega+n}$ -projective p -group A such that A/P is a direct sum of cyclic groups, it is necessary and sufficient that

(1) the non-zero elements in P have values $< \omega + n$;

(2) P is ω -fundamental.

Let A be a $p^{\omega+n}$ -projective p -group and P a p^n -bounded subgroup of A such that A/P is a direct sum of cyclics. Then the subgroup $p^\omega A$ of elements of infinite height is contained in P , hence $p^{\omega+n}A = 0$ and (1) follows. Since the heights of elements in $p^\omega A$ satisfy (2), the proof of necessity is complete.

Conversely, let P be a p^n -bounded group with valuation satisfying (1) and (2). For every $x \in P$ whose value is an integer n_x , select a fundamental cyclic group $\langle b_x \rangle$ of order p^{n_x} with $v(b_x) = n_x$, and for every $y \in P$ whose value is ω , choose infinitely many fundamental cyclic groups $\langle b_y^1 \rangle, \dots, \langle b_y^k \rangle, \dots$ of order p^n with $v(b_y^k) = k$. The direct sum $B = \bigoplus_x \langle b_x \rangle \oplus \bigoplus_{y,k} \langle b_y^k \rangle$ is then a free, valued p^n -bounded group such that the function $b_x \rightarrow x, b_y^k \rightarrow y$ extends to a morphism $\phi: B \rightarrow P$, and the natural map between $B/\text{Ker } \phi$ and P is an isometry; in fact, this follows at once from (1) and (2). For every b_x, b_y^k , select cyclic groups $\langle a_x \rangle, \langle a_y^k \rangle$ of orders p^{n+n_x} and p^{n+k} , respectively. Then under the correspondence $b_x \rightarrow p^{n_x} a_x, b_y^k \rightarrow p^k a_y^k$, B can be identified with a subgroup of $F = \bigoplus_x \langle a_x \rangle \oplus \bigoplus_{y,k} \langle a_y^k \rangle$ such that the heights of elements of B in F are given by the above valuation of B . Then $A = F/\text{Ker } \phi$ will be $p^{\omega+n}$ -projective whose quotient mod $B/\text{Ker } \phi$ is a direct sum of cyclic groups.

3. Let A be a $p^{\omega+n}$ -projective p -group and P a p^n -bounded subgroup of A such that A/P is a direct sum of cyclic groups. This P is not uniquely determined. But if we have another p^n -bounded P' in A with A/P' a direct sum of cyclics, then $P \cap P'$ has also the property that $A/(P \cap P')$ is a direct sum of cyclics (since it is a subgroup of $(A/P) \oplus (A/P')$). In order to learn a bit more about these P , we introduce the following concept.

Let S be a p -group with valuation. We will call S *distinctive* if there is a monomorphism of S into a direct sum of cyclic p -groups which does not decrease valuation. Keeping the notations of the preceding paragraph, we see that both $P/(P \cap P')$ and $P'/(P \cap P')$ are distinctive.

The following theorem records the most relevant fact we need in our discussion of distinctive p -groups.

Theorem 2. Let G be a p -group and S a subgroup of G such that G/S is a direct sum of cyclic groups. Let S be equipped with the valuation given by the heights in G . If S is distinctive, then G is a direct sum of cyclic groups.

Let $\psi: S \rightarrow C$ be a homomorphism of S into a direct sum of cyclics C which does not decrease heights. There is a group H and a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & S & \rightarrow & G & \rightarrow & G/S \rightarrow 0 \\ & & \downarrow \psi & & \downarrow \psi^* & & \parallel \\ 0 & \rightarrow & C & \rightarrow & H & \rightarrow & G/S \rightarrow 0 \end{array}$$

where $\psi^*: G \rightarrow H$ is monic (ψ being monic). It is easy to check that the second row is pure-exact. Since G/S is pure-projective ($= p^\omega$ -projective), the second row splits. Thus H is a direct sum of cyclics, and so are its subgroups. Consequently, G is a direct sum of cyclics, indeed.

By making use of a corollary to a rather deep theorem by P. HILL (see e.g. [2], Theorem 81.4), we can give another proof. Since S is closed in the p -adic topology of G , it is nice in G , and since G/S is totally projective, the cited theorem guarantees that ψ extends to a homomorphism $\varphi: G \rightarrow C$. This together with the natural homomorphism $G \rightarrow G/S$ yields an embedding $G \rightarrow C \oplus G/S$, a direct sum of cyclics.

As an immediate corollary, we can state:

Corollary 1. *Suppose A is a $p^{\omega+n}$ -projective p -group and P is a p^n -bounded subgroup with A/P a direct sum of cyclics. If P_0 is a subgroup of P such that P/P_0 is distinctive, then A/P_0 is again a direct sum of cyclic groups.*

Using the same idea, we can answer the question as to when a $p^{\omega+n}$ -projective p -group is properly $p^{\omega+n}$ -projective.

Theorem 3. *Let A be a $p^{\omega+n}$ -projective p -group and P a p^n -bounded subgroup of A such that A/P is a direct sum of cyclic groups. A is $p^{\omega+n-1}$ -projective if and only if P contains a p^{n-1} -bounded subgroup P' such that P/P' is distinctive.*

Sufficiency is a simple consequence of Corollary 1. To prove necessity, let Q be a p^{n-1} -bounded subgroup of A with A/Q a direct sum of cyclics. Then P/P' with $P' = P \cap Q$ is evidently distinctive.

4. We turn our attention to the structure of $p^{\omega+n}$ -projective p -groups. Recall that the σ th relative invariant $f_\sigma(G, A)$ of a subgroup G of a p -group A is defined as the dimension of the quotient

$$p^\sigma A[p] / ((p^{\sigma+1}A + G) \cap p^\sigma A[p]),$$

viewed as a vector space over the integers mod p (cf. Hill [4]; see also [2]).

The following lemma is fundamental.

Lemma. *Let G be a p^n -bounded subgroup of a p -group A . Then the relative invariants $f_\sigma(G, A)$ of G in A can be computed by using only the elements of $A[p^n]$.*

We can rewrite $(p^{\sigma+1}A + G) \cap p^\sigma A[p]$ as $(p^{\sigma+1}A + G)[p] \cap p^\sigma A[p]$. Our assertion will follow at once if we can show that

$$(p^{\sigma+1}A + G)[p] = (p^{\sigma+1}A[p^n] + G)[p].$$

But if $p(a+g)=0$ where $a \in p^{\sigma+1}A$ and $g \in G$, then $p^n a = -p^n g \in p^n G = 0$ implies $a \in p^{\sigma+1}A[p^n]$.

It is now easy to prove the next result which is stated in a more general fashion than needed for our purposes.

Theorem 4. *Let A and C be p -groups. Suppose that*

- (i) *there are p^n -bounded nice subgroups, P and Q , in A and C , respectively, such that A/P and C/Q are totally projective (or, moreover, direct sums of cyclics);*
- (ii) *there is a height-preserving isomorphism*

$$\varphi: A[p^n] \rightarrow C[p^n]$$

such that $\varphi P = Q$.

Then $A \cong C$.

Since P, Q are p^n -bounded and φ is height-preserving, from Lemma it follows at once that the relative invariants of P in A are equal to those of Q in C . Therefore, we have a height-preserving isomorphism $\varphi|_P$ between the nice subgroups P and Q with totally projective quotients $A/P, C/Q$ and the same relative invariants. The existence of an isomorphism $A \rightarrow C$ (extending $\varphi|_P$) follows at once from Hill's theorem [4] (cf. also [2, Theorem 83.4]).

Our main objective is to improve on Theorem 4 in the special case when A/P and C/Q are direct sums of cyclic groups. Observe that in this case P and Q are automatically nice subgroups (for subgroups closed in the p -adic topology are always nice). We want to show that even the hypothesis $\varphi P = Q$ can be removed.

Theorem 5. *Two $p^{\omega+n}$ -projective p -groups, A and C , are isomorphic precisely if there is a height-preserving isomorphism between $A[p^n]$ and $C[p^n]$.*

The necessity being obvious, suppose $\varphi: A[p^n] \rightarrow C[p^n]$ is a height-preserving isomorphism. Since A and C are $p^{\omega+n}$ -projective p -groups, there exist p^n -bounded subgroups P in A and Q in C such that A/P and C/Q are direct sums of cyclic p -groups. Consider the quotient group $A/(P \cap \varphi^{-1}Q)$ qua an extension of $P/(P \cap \varphi^{-1}Q)$ by A/P . The map

$$x + (P \cap \varphi^{-1}Q) \mapsto \varphi x + Q \quad (x \in P)$$

of $P/(P \cap \varphi^{-1}Q)$ into C/Q is monic, and it evidently does not decrease heights. Application of Theorem 2 leads us to conclude that $A/(P \cap \varphi^{-1}Q)$ is again a direct sum of cyclic groups. For reasons of symmetry, $C/(\varphi P \cap Q)$ is likewise a direct sum of cyclic groups. We are now in the situation of Theorem 4, with $P \cap \varphi^{-1}Q$ and $\varphi P \cap Q$ playing the roles of P and Q , respectively. Consequently, A and C are isomorphic, as claimed.

In other words, Theorem 5 states that $p^{\omega+n}$ -projective p -groups are completely characterized by their p^n -socles if regarded as groups with valuations.

References

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