Close packing and loose covering with balls

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To the memory of A. Kertész

The problem of the densest packing of balls, as well as the problem of the thinnest covering with balls have a vast literature [1, 2]. In this paper we want to call the attention to a variant of these problems which seems to offer ample scope for work.

Problem 1. In a space of constant curvature let P be a packing of balls of radius r. Let $\varrho=\varrho(P)$ be the supremum of the radii of those balls which have no point in common with any ball of P. Find the infimum $\bar{\varrho}=\bar{\varrho}(r)$ of ϱ extended over all packings P of balls of radius r.

Problem 2. In a space of constant curvature let C be a covering of balls of radius R. Let $\mathsf{P}=\mathsf{P}(C)$ be the supremum of the radii of those balls which are contained in the intersection of two balls of C. Find the infimum $\bar{\mathsf{P}}=\bar{\mathsf{P}}(R)$ of P extended over all coverings C with balls of radius R.

We call a packing with $\varrho=\bar{\varrho}$ a closest packing, in short a *close packing* and a covering with $\mathsf{P}=\bar{\mathsf{P}}$ a loosest covering, in short a *loose covering*. In certain special cases, as for instance in spherical spaces or in the Euclidean plane, the existence of a close packing and a loose covering with equal balls is obvious. But in Euclidean n-space with $n>2$ the question of existence seems to be difficult. Apart from the "regular" cases the same can be said about hyperbolic n-space with $n>1$.

In order to avoid a separate discussion of some uninteresting cases, we shall mean by a spherical ball only a ball not greater than a half-space, i.e. a ball of radius $\leqq\pi/2$.

If we have a packing of balls of radius r then concentric balls of radius $R=r+\varrho$ cover the space. Similarly, if a set of balls with radius R cover the space then concentric balls with radius $r=R-\mathsf{P}$ will form a packing. Thus, completing with the question of existence, the above problems can be summarized as follows: In the (r, R)-plane find the set of points such that balls of radius r form a packing and concentric balls of radius R form a covering.

Let us scrutinize this problem in spherical 2-space. Here the points (r_i, R_i) $i=1, 2, 3$ will play a special part, where r_1, r_2, r_3 are the inradii and R_1, R_2, R_3 are the circumradii of a face of the tessellation $\{5, 3\}$, $\{4, 3\}$ and $\{3, 3\}$, respectively.

Let the unit sphere be packed with n circles $c_1, \ldots, c_n$ of radius r and covered with concentric circles $C_1, \ldots, C_n$ of radius R. Let $D_1, \ldots, D_n$ be the Dirichlet cells of the centers. If p_i is the number of sides of D_i then, as a well known consequence of Euler's polyhedron theorem,

$$p_1+\ldots+p_n \leqq 6n-12$$

with equality only if the Dirichlet cells form a trihedral tessellation. Therefore there is among the Dirichlet cells one, say, D_i which has at most five sides. Since $c_i \subset D_i \subset C_i$, it follows that C_i cannot be smaller than the circumcircle of a regular pentagon circumscribed about c_i. Therefore $\tan R \geqq \tan r/\cos 36^\circ$.

It is known (see e.g. [1]) that the number n of circles of radius $r > r_1$ which can be packed on the sphere is less than 12. But for $n < 12$ the above inequality for the p_i's implies that there is a p_i less than five. Thus for $r > r_1$ C_i cannot be smaller than the circumcircle of a regular quadrangle circumscribed about c_i, i.e. $\tan R \geqq \tan r/\cos 45^\circ$.

Now we refer to the fact that at most four circles of radius $> r_2$, and at most three circles of radius $> r_3$ can be packed on the sphere. On the other hand, the radius of four circles covering the sphere is at least R_3, and the radius of three circles covering the sphere is $\pi/2$. Thus for $r > r_2$ we have $R \geqq R_3$, and for $r > r_3$ we have $R = \pi/2$.

To sum up we phrase the following

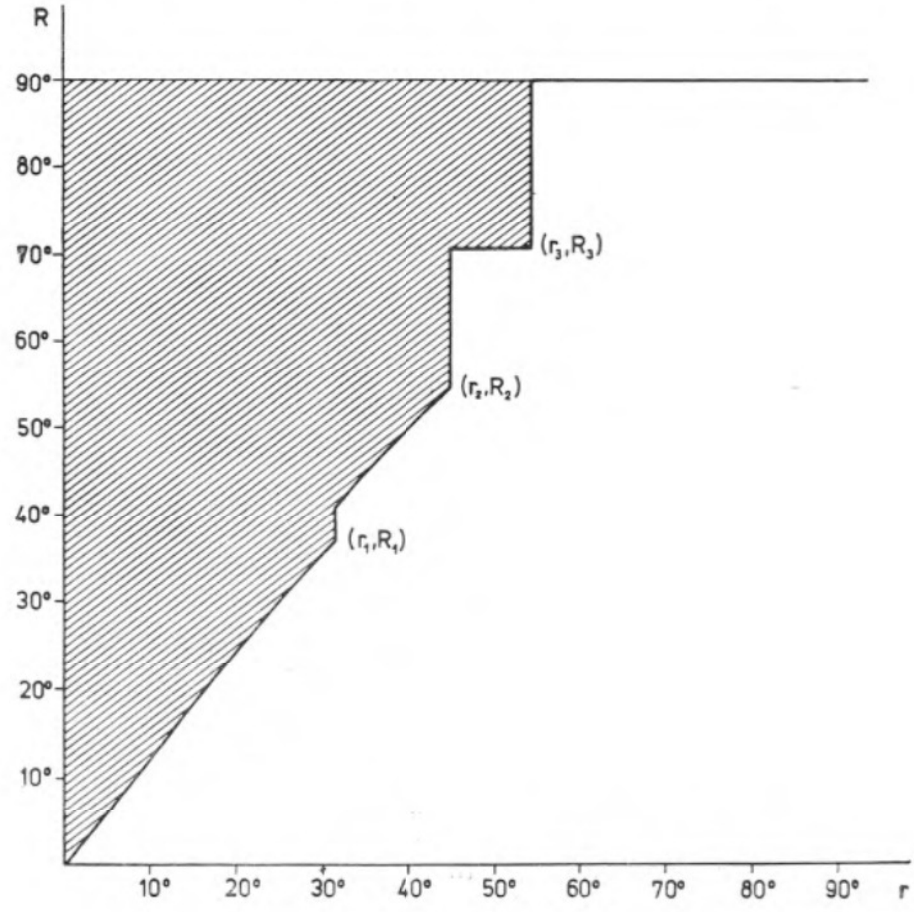

Fig. 1.

Theorem. *If the sphere is packed with at least two circles of radius r and covered with concentric circles of radius R then we have*

$$\frac{\tan r}{\tan R} \leqq \begin{cases} \cos 36^\circ & \text{for } 0 < r \leqq r_1 \\ \cos 45^\circ & \text{for } r_1 < r \leqq r_2. \end{cases}$$

For $r_2 < r \leqq r_3$ we have $R \geqq R_3$ and for $r_3 < r$ we have $R = \pi/2$.

These bounds are represented in *Fig.* 1.

It is interesting to observe, that, apart from the regular cases corresponding to the points (r_i, R_i) $(i=1, 2, 3)$ and the cases with $r > r_2$, equality can be attained also in several other cases. Let $ABCD$ be a regular spherical quadrangle centered at the northpole N such that the images T, U, V and W of N reflected in the sides AB, BC, CD and DA, respectively, are the vertices of a quadrangle congruent to $ABCD$. Adding to the points $N, A, \dots, W$ the southpole S we obtain the vertices of an antiprismatic doublepyramid [3]. This solid is bounded by 16 equal isosceles triangles one of which is NAB. Since in the spherical triangle NAB $\sphericalangle A = \sphericalangle B = 270^\circ/4 < \sphericalangle N = 90^\circ$, we have $AB > NA = NB$. Therefore circles of radius $r = NA/2$ centered at the vertices of the solid will form a packing. In this packing the Dirichlet cells belonging to N and S are regular quadrangles (*Fig.* 2) circumscribed about the respective circles

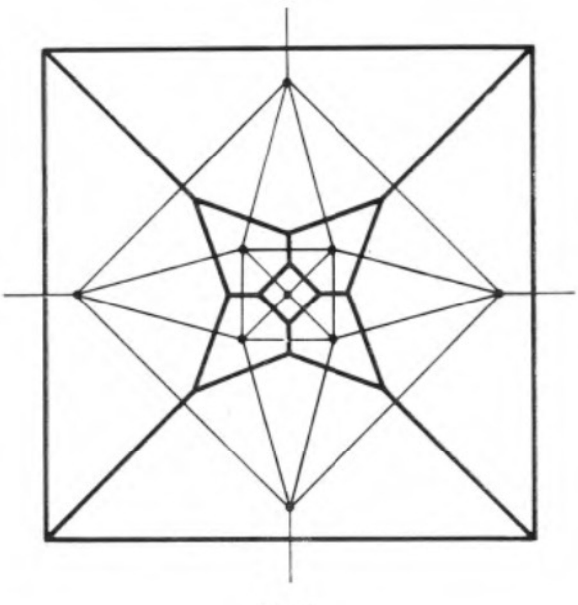

Fig. 2.

On the other hand, the radius R of the circle circumscribed about one of these quadrangles is nothing else as the circumradius of NAB, showing that the circles of radius R with centers $N, A, \dots, S$ cover the sphere. Since r and R are the inradius and circumradius of a regular quadrangle, we have $\tan R = \tan r / \cos 45^\circ$.

As a second example consider the set S of 32 points consisting of the vertices and face-centers of the tessellation $\{5, 3\}$. The Dirichlet cells are regular pentagons concentric with the faces of $\{5.3\}$ and (not regular) hexagons about the vertices of

$\{5, 3\}$. For the inradius r and circumradius R of the pentagons we have $\tan R = \tan r/\cos 36°$. Since, on the other hand, the hexagons have the same inradius and circumradius as the pentagons, the circles of radius r and R about the points of S form a packing and a covering, respectively.

The question whether there are further cases with $r < r_2$ in which equality is attained is still open.

If we have at least three circles then $r \leqq 60°$. Throwing a glance to Fig. 1 we see that the set of admissible points (r, R) with $r \leqq 60°$ lies above the half-line connecting the origin $(0, 0)$ with the point (r_1, R_1). Thus we have the following

Corollary. *If the sphere is packed with at least three circles of radius r and covered with concentric circles of radius R then* $R/r \geqq R_1/r_1$.

To conclude we mention some further problems.

In Euclidean 3-space an interesting problem seems to be to find the closest lattice-packing of balls. It is very likely that in this packing the centers form a space-centered cubic lattice. This would mean that the loosest lattice-covering is identical with the thinnest lattice-covering.

We can define a closest packing of convex bodies as a packing in which the "biggest gap-ball" is as small as possible. Measuring the closeness of a packing with the curvature of the biggest gap-ball we can ask various questions similar to those which arise in connection with the density. For instance, is it true that in the Euclidean plane the closeness of a packing of equal centro-symmetric convex plates cannot exceed the closeness of the closest lattice-packing of the plates?

In a packing of translates of a convex plate p of area A we can measure the closeness also by the quotient A/a, where a is the supremum of the area of those plates homothetic to p which have no point in common with any plate of the packing. It may be conjectured that in a closest packing in this sense we have $A/a \leqq 16$ with equality only if p is a triangle.

References

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