

On regular polynomial matrices II

The associated matrix pencils; simple matrix pencils

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1. Introduction

Our paper [1] was devoted to regular polynomial matrices giving also a new criterion for a polynomial matrix to be simple.

The present paper is composed of two parts. In part one we will show some relations existing between a regular polynomial matrix $D_l(\lambda, \mu)$ and the associated matrix pencil. The second part is a supplement to [1], since in the special case of simple matrix pencils some statements of [1] can be formulated sharper.

The following terms and notations will be used. A regular polynomial matrix $D_l(\lambda, \mu)$ of degree l will always denote its homogeneous form

$$(1.1) \quad D_l(\lambda, \mu) = A_0 \lambda^l + A_1 \lambda^{l-1} \mu + \dots + A_l \mu^l; \quad A_0 \neq 0; \quad |D_l(\lambda, \mu)| \neq 0,$$

where all A_i are n -square complex matrices. The characteristic polynomial of $D_l(\lambda, \mu)$ is

$$(1.2) \quad \Delta(\lambda, \mu) \doteq |D_l(\lambda, \mu)| = c \mu^{\alpha_0} \prod_{k=1}^s (\lambda - \lambda_k \mu)^{\alpha_k}$$

$$c \neq 0; \quad \lambda_k \neq \lambda_i \Leftrightarrow k \neq i; \quad \sum_{k=0}^s \alpha_k = nl.$$

The reduced characteristic polynomial

$$(1.3) \quad \delta(\lambda, \mu) \doteq \frac{\Delta(\lambda, \mu)}{\Delta_{n-1}(\lambda, \mu)} = c \mu^{\beta_0} \prod_{k=1}^s (\lambda - \lambda_k \mu)^{\beta_k}$$

$\lambda_k \neq \lambda_i \Leftrightarrow k \neq i; \quad 1 \leq \beta_k \leq \alpha_k \quad (k = 1, \dots, s), \quad 1 \leq \beta_0 \leq \alpha_0 \text{ if } \alpha_0 \neq 0.$ *)
and the reduced adjoint of $D_l(\lambda, \mu)$

$$(1.4) \quad F(\lambda, \mu) \doteq \frac{\text{adj } D_l(\lambda, \mu)}{\Delta_{n-1}(\lambda, \mu)}$$

play an important role in our discussions. Here $\Delta_{n-1}(\lambda, \mu)$ denotes the gcd of all $(n-1)$ th order subdeterminants of $D_l(\lambda, \mu)$.

*) The inequality $1 \leq \beta_k \leq \alpha_k$ is proved in [1].

The roots of an arbitrary polynomial $f(\lambda, \mu)$ will always mean the — finite and infinite — roots of the polynomial equation $f(\lambda, \mu)=0$. For any homogeneous polynomial $f(\lambda, \mu)$ we will use the abbreviation

$$(1.5) \quad f(\lambda_k) = \begin{cases} f(\lambda, \mu)|_{\lambda=\lambda_k, \mu=1} & k = 1, \dots, s \\ f(\lambda, \mu)|_{\lambda=1, \mu=0} & k = 0 \end{cases}$$

$$f^{(v)}(\lambda_k) = \begin{cases} \left. \frac{\partial^v f(\lambda, \mu)}{\partial \lambda^v} \right|_{\lambda=\lambda_k, \mu=1} & k = 1, \dots, s \\ \left. \frac{\partial^v f(\lambda, \mu)}{\partial \mu^v} \right|_{\lambda=1, \mu=0} & k = 0, \end{cases}$$

where λ_k ($k=0, 1, \dots, s$) are the roots of $\delta(\lambda, \mu)$ in (1.3).

The asterisk on a matrix (or column vector) indicates the conjugate transpose; E_r is the r -square unit matrix; $\langle d_0, d_1, \dots, d_s \rangle$ denotes the diagonal matrix with d_k in the k th position of the main diagonal; $\varrho(X)$ is the rank of the matrix X .

A regular polynomial matrix will always mean a polynomial matrix represented by (1.1)—(1.3). A regular polynomial matrix $D_l(\lambda, \mu)$ is called *simple* if $\varrho[D_l(\lambda_k)] = n - \alpha_k$ ($k=0, 1, \dots, s$) (cf. [1]). A regular matrix pencil is a linear binomial matrix

$$(1.6) \quad \lambda B + \mu A; \quad |\lambda B + \mu A| \not\equiv 0.$$

2. The associated matrix pencil

To a regular polynomial matrix $D_l(\lambda, \mu)$ there can be associated a regular matrix pencil in several ways (cf. [2], [3], [4]). We associate to $D_l(\lambda, \mu)$ the matrix pencil $\lambda \mathcal{B} + \mu \mathcal{A}$, where $\mathcal{B}$ and $\mathcal{A}$ are ln -square matrices in the partitioned form

$$(2.1) \quad \mathcal{B} = \begin{bmatrix} A_0 & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{bmatrix}; \quad \mathcal{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_{l-1} & A_l \\ -E & & & & \\ & \ddots & & & \\ & & & -E & 0 \end{bmatrix}.$$

In the following some relations between $D_l(\lambda, \mu)$ and the associated pencil $\lambda \mathcal{B} + \mu \mathcal{A}$ will be verified.

Theorem 2.1. *Let $D_l(\lambda, \mu)$ be a regular polynomial matrix and $\lambda \mathcal{B} + \mu \mathcal{A}$ the associated matrix pencil; then*

- (i) $|D_l(\lambda, \mu)| \equiv |\lambda \mathcal{B} + \mu \mathcal{A}|,$
- (ii) $\varrho[D_l(\lambda_k)] = \varrho(\lambda_k \mathcal{B} + \mathcal{A}) - n(l-1)$
 $k = 0, 1, \dots, s; \quad (\lambda_0 \mathcal{B} + \mathcal{A}) \doteq \mathcal{B},$
- (iii) $D_l(\lambda, \mu)$ and $\lambda \mathcal{B} + \mu \mathcal{A}$ have the same elementary divisors.

PROOF. It can be verified by means of direct calculation that the identity

$$(2.2) \quad \mathcal{S}(\lambda\mathcal{B} + \mu\mathcal{A})\mathcal{T} = \begin{bmatrix} D_l(\lambda, \mu) & \\ & E_{n(l-1)} \end{bmatrix}$$

holds, where S and T are the partitioned matrices

$$(2.3) \quad \mathcal{S} = \begin{bmatrix} E & \mu^{-1}(\lambda A_0 + \mu A_1) & A_2 & \dots & A_{l-2} & A_{l-1} \\ & E & \mu^{-1}\lambda E & & & \\ & & \ddots & \ddots & & \\ & & & E & & \\ & & & & \mu^{-1}\lambda E & \\ & & & & & E \end{bmatrix}$$

and

$$(2.4) \quad \mathcal{T} = \begin{bmatrix} E & & \lambda^{l-1}E \\ & E & \lambda^{l-2}\mu E \\ & & \ddots \\ & & & E & \lambda\mu^{l-2}E \\ & & & & \mu^{l-1}E \end{bmatrix} \begin{bmatrix} 0 & -\mu^{-1}E \\ & 0 & -\mu^{-1}E \\ & & \ddots & \ddots \\ & & & -\mu^{-1}E \\ E & & & & 0 \end{bmatrix}.$$

The determinant of S and T is according to (2.3) and (2.4)

$$|S| = 1, \quad |T| = 1,$$

hence (2.1) is a *strictly equivalent relation*. Assertion (i)—(iii) follows immediately from (2.2) taking into account (2.3) and (2.4).

Remark. We know already from (i) and (ii) of theorem 2.1 that

(i') $D_l(\lambda, \mu)$ and $\lambda\mathcal{B} + \mu\mathcal{A}$ have the same eigenvalues taken into account with their multiplicities,

(ii') $D_l(\lambda, \mu)$ has exactly as many linearly independent eigenvectors — belonging to the eigenvalue λ_k of multiplicity α_k — as $\lambda\mathcal{B} + \mu\mathcal{A}$.

Therefore we obtain from (i) and (ii) the following theorem, its formulation agreeing with that of theorem 4.2 of LANCASTER [4].

Theorem 2.2. *A regular polynomial matrix $D_l(\lambda, \mu)$ is simple if and only if the associated pencil $\lambda\mathcal{B} + \mu\mathcal{A}$ is a simple pencil.*

The following theorem can be verified by simple calculation.

Theorem 2.3. Let $D_l(\lambda, \mu)$ be a regular polynomial matrix. If u_k is a right eigenvector of $D_l(\lambda, \mu)$ belonging to the eigenvalue λ_k , then the partitioned vector

$$\begin{aligned} & [\lambda_k^{l-1} u_k^*, \lambda_k^{l-2} u_k^*, \dots, \lambda_k u_k^*, u_k^*]^* \quad k = 1, \dots, s \\ & [u_0^*, 0, \dots, 0, 0]^* \quad k = 0 \end{aligned}$$

is a right eigenvector of the associated matrix pencil $\lambda B + \mu A$ belonging to the same eigenvalue λ_k . A similar statement holds for left eigenvectors as well.

3. Simple matrix pencils

All statements and relations established in [1] concerning regular polynomial matrices are valid of course also for regular matrix pencils; it is even true that some of the given expressions and relations receive a simpler form if formulated concerning regular matrix pencils, which are a very simple special case of regular polynomial matrices in general. In spite of this fact we will not go into details in reformulating the results of [1] concerning matrix pencils. Here we restrict ourselves to the formulation of additional relations valid for matrix pencils only. Our starting-point is theorem 1 of [1] which can be formulated concerning matrix pencils as follows:

Theorem 3.1. Let $\lambda B + \mu A$ be a regular matrix pencil. If the polynomial $\delta(\lambda, \mu)$ has only simple roots, then for the matrices

$$(3.1) \quad P_k = \frac{1}{\delta'(\lambda_k)} F(\lambda_k) \quad k = 0, 1, \dots, s$$

the following assertions hold:

- (i) $Q(P_k) = Q(BP_k) = Q(P_k B) = \alpha_k \quad k = 1, \dots, s$
 $Q(P_0) = Q(AP_0) = Q(P_0 A) = \alpha_0 \quad k = 0$
- (ii) $(BP_k)^2 = BP_k; \quad (P_k B)^2 = P_k B \quad k = 1, \dots, s$
 $(AP_0)^2 = AP_0; \quad (P_0 A)^2 = P_0 A \quad k = 0$
- (iii) $P_i A P_k = P_i B P_k = 0 \quad \text{if } i \neq k.$

PROOF. Assertions (i) and (ii) agree with those of theorem 1 of [1] considering that for matrix pencils

$$\frac{\partial(\lambda B + \mu A)}{\partial \lambda} = B; \quad \frac{\partial(\lambda B + \mu A)}{\partial \mu} = A.$$

According to (3.1) it is sufficient to prove (iii) for the matrices $F(\lambda_k)$ and $F(\lambda_i)$ $i \neq k$. We consider the relations

$$\begin{aligned} (\lambda B + \mu A) F(\lambda, \mu) &= \delta(\lambda, \mu) E \\ F(\lambda, \mu) (\lambda B + \mu A) &= \delta(\lambda, \mu) E. \end{aligned}$$

Substituting $\lambda = \lambda_k$ and $\lambda = \lambda_i$, resp. ($k \neq i$), we obtain

$$(3.2) \quad \begin{aligned} \lambda_k B F(\lambda_k) &= -A F(\lambda_k) \\ \lambda_i F(\lambda_i) B &= -F(\lambda_i) A. \end{aligned}$$

Premultiplying the first relation of (3.2) by $F(\lambda_i)$ and postmultiplying the second by $F(\lambda_k)$, we obtain

$$(3.3) \quad (\lambda_k - \lambda_i) F(\lambda_i) B F(\lambda_k) = 0$$

and since $\lambda_k \neq \lambda_i$, it follows according to (3.2) and (3.3) that

$$F(\lambda_i) B F(\lambda_k) = F(\lambda_i) A F(\lambda_k) = 0 \quad i \neq k; \quad i, k \neq 0.$$

If in (3.2) we premultiply the first relation and postmultiply the second by $F(\lambda_0)$, we obtain

$$F(\lambda_0) A F(\lambda_k) = F(\lambda_0) B F(\lambda_k) = F(\lambda_k) B F(\lambda_0) = F(\lambda_k) A F(\lambda_0) = 0$$

since $B F(\lambda_0) = F(\lambda_0) B = 0$. Assertion (iii) is proved now for all cases.

Let the matrices (3.1) be decomposed in basis factors, that is

$$(3.4) \quad P_k = U_k V_k^*; \quad \varrho(U_k) = \varrho(V_k) = \varrho(P_k) \quad k = 0, 1, \dots, s,$$

where U_k and V_k are matrices of type $n \times \alpha_k$. It is known from [1] that for $\lambda B + \mu A$ the columns of U_k and the rows of V_k^* yield α_k linearly independent right and left eigenvectors resp., belonging to the eigenvalue λ_k of multiplicity α_k . Hence we obtain from theorem 3.1.

Corollary 1. If $\delta(\lambda, \mu)$ has only simple roots, then the regular matrix pencil $\lambda B + \mu A$ is simple and can be transformed into the diagonal form

$$V^*(\lambda B + \mu A)U = \langle \mu E_{\alpha_0}, (\lambda - \lambda_1 \mu) E_{\alpha_1}, \dots, (\lambda - \lambda_s \mu) E_{\alpha_s} \rangle,$$

or by putting $\mu = 1$,

$$V^*(\lambda B + A)U = \langle E_{\alpha_0}, (\lambda - \lambda_1) E_{\alpha_1}, \dots, (\lambda - \lambda_s) E_{\alpha_s} \rangle.$$

The transformation matrices V^* and U are obtained from the matrices in (3.4) as

$$V^* = [V_0 V_1 \dots V_s]^*; \quad U = [U_0 U_1 \dots U_s].$$

It is of interest to note here that according to theorem 3 of [1] the following two statements are equivalent for a regular matrix pencil $\lambda B + \mu A$:

- $\alpha)$ $\lambda B + \mu A$ is simple,
- $\beta)$ $\delta(\lambda, \mu)$ has only simple roots.

It is stated in theorem 1 of [1] that under the assumption of $\delta(\lambda, \mu)$ having only simple roots, $D'_i(\lambda_k) P_k$ and $P_k D'_i(\lambda_k)$ are idempotent matrices of rank α_k ($k = 0, 1, \dots, s$). In the case of matrix pencils these idempotents are $A P_0$, $B P_k$ ($k = 1, \dots, s$) and $P_0 A$, $P_k B$ ($k = 1, \dots, s$) resp., and according to (iii) of the theorem proved above *both sets of matrices are orthogonal*. Hence taking into account the well-known theorem concerning orthogonal idempotents (cf. e.g. [5] pp.), we obtain

Corollary 2. If for the regular matrix pencil $\lambda B + \mu A$ the polynomial $\delta(\lambda, \mu)$ has only simple roots, then

$$(3.5) \quad \begin{aligned} AP_0 + B \sum_{k=0}^s P_k &= E \\ P_0 A + \sum_{k=1}^s P_k B &= E. \end{aligned}$$

Before finishing this section we consider the restriction $|B| \neq 0$. Then (3.5) reduces to

$$B \sum_{k=1}^s P_k = E; \quad \sum_{k=1}^s P_k B = E$$

and this yields according to (3.2) the relations

$$B^{-1} = \sum_{k=1}^s P_k; \quad -A = B \sum_{k=1}^s \lambda_k P_k B.$$

Also the matrices (3.1) can be expressed in another form. There can be established

Theorem 3.2. Let $\lambda B + A$ ($|B| \neq 0$) be a simple matrix pencil, i.e. let the polynomial $\delta(\lambda)$ have only simple roots. If

$$L_k(\lambda) = \frac{\delta(\lambda)}{\delta'(\lambda_k)(\lambda - \lambda_k)} \quad k = 1, \dots, s$$

is the Lagrangean interpolation polynomial belonging to $\delta(\lambda)$, then the matrices (3.1) can be expressed as

$$(3.6) \quad P_k = L_k(-B^{-1}A)B^{-1}; \quad P_k = B^{-1}L_k(-AB^{-1}) \quad k = 1, \dots, s.$$

PROOF. It is sufficient to prove one form of (3.6). Let us introduce the notation

$$G_k^{(v)} = \frac{1}{(v+1)!} F^{(v)}(\lambda_k) \quad v = 0, 1, \dots; \quad k = 1, \dots, s.$$

It is easy to verify the relations

$$BG_k^{(v)} = \frac{\delta^{(v+1)}(\lambda_k)}{(v+1)!} E - (\lambda_k B + A)G_k^{(v+1)} \quad v = 0, 1, \dots, s.$$

Taking into account that $G_k^{(s-1)} = \frac{\delta^{(s)}(\lambda_k)}{s!} B^{-1} \neq 0$, we obtain successively

$$\begin{aligned} G_k^{(s-2)} &= \frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!} B^{-1} - B^{-1}(\lambda_k B + A) \frac{\delta^{(s)}(\lambda_k)}{s!} B^{-1} \\ &= \left\{ \frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!} E + (-B^{-1}A - \lambda_k E) \frac{\delta^{(s)}(\lambda_k)}{s!} E \right\} B^{-1} \\ G_k^{(s-3)} &= \left\{ \frac{\delta^{(s-2)}(\lambda_k)}{(s-2)!} E + (-B^{-1}A - \lambda_k E) \left[\frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!} E + (-B^{-1}A - \lambda_k E) \frac{\delta^{(s)}(\lambda_k)}{s!} E \right] \right\} B^{-1} \\ &\vdots \end{aligned}$$

and finally we arrive at

$$(3.7) \quad F(\lambda_k) = G_k^{(0)} = \left\{ \frac{\delta'(\lambda_k)}{1!} E + (-B^{-1}A - \lambda_k E) \left[\dots + (-B^{-1}A - \lambda_k E) \left[\frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!} E + \right. \right. \right. \\ \left. \left. \left. + (-B^{-1}A - \lambda_k E) \frac{\delta^{(s)}(\lambda_k)}{s!} E \right] \dots \right] \right\} B^{-1}.$$

The expression in the parenthesis $\{\dots\}$ on the right side of (3.7) is the Horner-arranging of the polynomial

$$(3.8) \quad \frac{\delta(\lambda)}{(\lambda - \lambda_k)} = \frac{\delta'(\lambda_k)}{1!} + \frac{\delta''(\lambda_k)}{2!} (\lambda - \lambda_k) + \dots + \frac{\delta^{(s)}(\lambda_k)}{s!} (\lambda - \lambda_k)^{s-1}$$

for $\lambda = -B^{-1}A$. Hence by (3.7) and (3.8) we have

$$P_k = \frac{1}{\delta'(\lambda_k)} F(\lambda_k) = \frac{\delta(-B^{-1}A)}{\delta'(\lambda_k)(-B^{-1}A - \lambda_k E)} B^{-1} = L_k(-B^{-1}A) B^{-1} \quad k = 1, \dots, s,$$

which proves the theorem.

Remark. In the case of $B=E$ the topic discussed in [1] and in this section reduces to the usual eigenvalue/eigenvector problem of the matrix $-A$. It is of interest to point to the fact, that in this most special case our approach followed in [1] and in this section agrees with that of the known approach of the eigenvalue/eigenvector problem of matrices of simple structure (see e.g. WEDDERBURN [6] and EGERVÁRY [7]).

References

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