

A sine functional equation in Banach algebras

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The functional equation $f(x+y+\alpha)-f(x-y+\alpha)=2f(x)f(y)$, where $\alpha \in R$ (the field of real numbers) was essentially first considered by E. B. VAN VLECK (cf. [1], [11]) for the case when $f: R \rightarrow R$ is continuous. The aim of this paper is to investigate this equation in the cases when $f: R \rightarrow B$, where B is a complex algebra, or Banach algebra, or the algebra of the bounded linear operators in a complex Banach space X . In the two latter cases measurability conditions will be imposed, which also ensure continuity of the solutions.

If B is a complex algebra, then the general solution is given in terms of its odd component, which on the other hand, is in connection with a type of exponential function. In the case of a complex Banach algebra B the concept of the generator element is defined and characterized, and the strongly measurable solutions are given in terms of the generator.

If $B=B(X)$ is the algebra of bounded linear operators in a complex Banach space X , then essentially continuity of the solutions in the strong operator topology is assumed and similar problems are investigated. Theorem 8 deals with the case when $X=H$ is a Hilbert space. Note that sometimes we use the terminology of [6] without explicitly mentioning this.

1. The equation in algebras and Banach algebras

Definition. Let R be the real field, B an algebra over the complex field, $S: R \rightarrow B$ a map, $\alpha \in R$. We shall say that $S \in V(\alpha)$, if for every $(\xi, \eta) \in R \times R$ we have

$$(1) \quad S(\xi + \eta + \alpha) - S(\xi - \eta + \alpha) = 2S(\xi)S(\eta)$$

We say that $S \in V_0(\alpha)$ if S is odd and satisfies (1).

Fundamental is in what follows the following

Theorem 1. If $S \in V_0(\alpha)$, then $S(-\alpha)=j$ is idempotent, and $jS(\xi)=S(\xi)j=S(\xi)$ holds for every $\xi \in R$. Putting $C(\xi)=S(\xi-\alpha)$, $C(\xi)$ satisfies the equation $C(\xi+\eta)+C(\xi-\eta)=2C(\xi)C(\eta)$ (d'Alembert's equation), and $E(\xi)=S(\xi-\alpha)+iS(\xi)$ satisfies the equations

$$(2) \quad \begin{aligned} E(\xi + \eta) &= E(\xi)E(\eta) \\ E(\xi + \alpha) &= -iE(\xi) \end{aligned}$$

Conversely, if $E: R \rightarrow B$ satisfies (2), then $S(\xi) = \frac{1}{2i} \{E(\xi) - E(-\xi)\} \in V_0(\alpha)$.

PROOF. 1. S is odd, therefore $S(0)=0$. Setting $\xi=0$ in (1), we have

$$(3) \quad S(\eta + \alpha) = S(-\eta + \alpha)$$

and hence

$$S(\eta)S(\xi) = \frac{1}{2} \{S(\xi + \eta + \alpha) - S(\eta - \xi + \alpha)\} = S(\xi)S(\eta) \quad \text{for } \xi, \eta \in R.$$

Since S is odd, putting $\xi = -\alpha$ in (1), we obtain that $S(\eta) = S(-\alpha)S(\eta)$ ($\eta \in R$) and $S(-\alpha) = j$ is idempotent. For the same reason (3) yields

$$(4) \quad S(\xi + \alpha) = -S(\xi - \alpha) \quad (\xi \in R)$$

and, considering (1) for $(\xi - \alpha, \eta - \alpha) \in R \times R$, we conclude that

$$(5) \quad S(\xi + \eta - \alpha) + S(\xi - \eta - \alpha) = 2S(\xi - \alpha)S(\eta - \alpha)$$

that is $C(\xi) = S(\xi - \alpha)$ satisfies d'Alembert's equation. In view of the definition of E

$$E(\xi)E(\eta) = C(\xi)C(\eta) - S(\xi)S(\eta) + i\{S(\xi)C(\eta) + C(\xi)S(\eta)\}.$$

Here we have $C(\xi)C(\eta) - S(\xi)S(\eta) = C(\xi)C(\eta) - C(\xi + \alpha)C(\eta + \alpha) = 1/2\{C(\xi + \eta) + C(\xi - \eta) - C(\xi + \eta + 2\alpha) - C(\xi - \eta)\} = C(\xi + \eta)$ according to (4) and (5). Similarly, by (1) and (4) we get

$$\begin{aligned} S(\xi)C(\eta) + C(\xi)S(\eta) &= S(\xi)S(\eta - \alpha) + S(\xi - \alpha)S(\eta) = \\ &= \frac{1}{2} \{S(\xi + \eta) - S(\xi - \eta + 2\alpha) + S(\xi + \eta) - S(\xi - \eta)\} = S(\xi + \eta) \end{aligned}$$

and thus $E(\xi)E(\eta) = E(\xi + \eta)$. Moreover, by (4)

$$E(\xi + \alpha) = C(\xi + \alpha) + iS(\xi + \alpha) = S(\xi) - iC(\xi) = -iE(\xi).$$

2. To prove the converse we observe that $E(\xi - \alpha) = iE(\xi)$ and

$$\begin{aligned} S(\xi + \eta + \alpha) - S(\xi - \eta + \alpha) &= \\ &= \frac{1}{2i} \{E(\xi + \eta + \alpha) - E(-\xi - \eta - \alpha) - E(\xi - \eta + \alpha) + E(-\xi + \eta - \alpha)\} = \\ &= \frac{1}{2i} \{-iE(\xi + \eta) - iE(-\xi - \eta) + iE(\xi - \eta) + iE(-\xi + \eta)\} = 2S(\xi)S(\eta). \end{aligned}$$

Since S obviously is odd, $S \in V_0(\alpha)$, and the proof is complete.

Definition. The functions C, E occurring in Theorem 1 will be called the functions C, E associated with $S \in V_0(\alpha)$.

Remark. If B_u is a subalgebra of B with unit u , $S: R \rightarrow B_u$, $S \in V(\alpha)$, and for some $\xi_0 \in R$ there exists the inverse of $S(\xi_0)$ in B_u (denoted by $S(\xi_0)_u^{-1}$), then $S \in V_0(\alpha)$. Indeed, by (1) we have for every $\eta \in R$

$$S(\xi_0)S(-\eta) = -S(\xi_0)S(\eta)$$

and, multiplying by $S(\xi_0)_u^{-1}$, we obtain that S is odd.

Theorem 2. *If $S \in V(\alpha)$, $S_e(\xi) = 1/2\{S(\xi) + S(-\xi)\}$, $S_0(\xi) = 1/2\{S(\xi) - S(-\xi)\}$, then $S_0 \in V_0(\alpha)$ and $S_e(\xi) = a S_0(\xi + \alpha) + a - aj$, where $a = S_e(2\alpha)$, $j = S_0(-\alpha)$ is idempotent, and $a^2 = 0$, $ja = 0$. Conversely, if $S_0 \in V_0(\alpha)$, $j = S_0(-\alpha)$ and there exists an $a \in B$ such that $a^2 = 0$, $ja = 0$, then $S(\xi)$ defined by $S_0(\xi) + a S_0(\xi + \alpha) + a - aj$ belongs to $V(\alpha)$ and S_0 is the odd part of S .*

PROOF. 1. If $S \in V(\alpha)$, then $S(\xi)S(-\eta) = -S(\xi)S(\eta)$ for $\xi, \eta \in R$, and hence

$$(6) \quad S(\xi)S_e(\eta) = 0 \quad (\xi, \eta \in R).$$

Putting the pair $(-\xi, \eta)$ in (6) we get

$$(6a) \quad S_e(\xi)S_e(\eta) = 0 \quad (\xi, \eta \in R)$$

$$(6b) \quad S_0(\xi)S_e(\eta) = 0$$

With $\xi = -\alpha$ in (1) we obtain

$$(7) \quad S(-\alpha)S(\eta) = \frac{1}{2} \{S(\eta) - S(-\eta)\} = S_0(\eta)$$

while (1) becomes because of (6)

$$(8) \quad S(\xi + \eta + \alpha) - S(\xi - \eta + \alpha) = 2S(\xi)S_0(\eta)$$

Multiplying (8) by $S(-\alpha)$ from the left, we have by (7)

$$(9) \quad S_0(\xi + \eta + \alpha) - S_0(\xi - \eta + \alpha) = 2S_0(\xi)S_0(\eta)$$

and thus $S_0 \in V_0(\alpha)$. Using (8) again, we obtain

$$(10) \quad S_e(\xi + \eta + \alpha) - S_e(\xi - \eta + \alpha) = 2S_e(\xi)S_0(\eta)$$

Considering (10) for the pair $(\xi - \alpha, \eta)$ implies

$$(11) \quad S_e(\xi - \alpha)S_0(\eta) = \frac{1}{2} \{S_e(\xi + \eta) - S_e(\xi - \eta)\} = S_e(\eta - \alpha)S_0(\xi) \quad (\xi, \eta \in R)$$

for the middle part is symmetric. From (9) it follows that $S_0(-\alpha) = j$ is idempotent, and (11) for the pair $(\xi + \alpha, -\alpha)$ gives that

$$(12) \quad S_e(\xi) \cdot j = S_e(2\alpha)S_0(\xi + \alpha).$$

Then for every $\xi, \eta \in R$ it follows that

$$(13) \quad S_e(\xi)S_0(\eta) = S_e(\xi) \cdot j \cdot S_0(\eta) = S_e(2\alpha)S_0(\xi + \alpha)S_0(\eta).$$

Putting $\eta = \xi - \alpha$ in (10) and making use of (13) gives

$$S_e(2\xi) = S_e(2\alpha) + 2S_e(2\alpha)S_0(\xi + \alpha)S_0(\xi - \alpha)$$

and by (9) we get

$$S_e(2\xi) = S_e(2\alpha) + S_e(2\alpha) \{S_0(2\xi + \alpha) - S_0(3\alpha)\}.$$

From (4) it can be seen that S_0 is periodic with period $4|\alpha|$, thus with the notation of the theorem we have $S_e(\xi) = a + a \{S_0(\xi + \alpha) - j\}$. Moreover, from (6a) and (6b) we see that $a^2 = 0$, $ja = 0$.

2. If $S_0 \in V_0(\alpha)$ and $S(\xi)$ is of the asserted form, then with the notation $S_e(\xi) = S(\xi) - S_0(\xi)$ we obtain that $S_0(\xi)S_e(\eta) = 0$ ($\xi, \eta \in R$), for $ja = 0$. Similarly, since $a^2 = 0$, it follows that $S_e(\xi) \cdot S_e(\eta) = \{a + a[S_0(\xi + \alpha)j - j]\} \cdot \{a + a[S_0(\eta + \alpha) - j]\} = 0$. Thus, using (9), we establish that $2S(\xi)S(\eta) = 2\{a + a[S_0(\xi + \alpha) - j] + S_0(\xi)\}S_0(\eta) = 2 \cdot S_0(\xi)S_0(\eta) + 2aS_0(\xi + \alpha)S_0(\eta) + 2aS_0(\eta) - 2ajS_0(\eta) = S_0(\xi + \eta + \alpha) - S_0(\xi - \eta + \alpha) + a\{S_0(\xi + \eta + 2\alpha) - S_0(\xi - \eta + 2\alpha)\} = S(\xi + \eta + \alpha) - S(\xi - \eta + \alpha)$, hence $S \in V(\alpha)$. The last statement is obvious, because in view of $S_0 \in V_0(\alpha)$ $S_0(-\xi + \alpha) = S_0(\xi + \alpha)$ holds.

Corollary 1. If $S \in V(\alpha)$ and $S(0) = 0$, then $S \in V_0(\alpha)$.

PROOF. Using the notations of the preceding theorem $S_0 \in V_0(\alpha)$, and (4) implies that $S_0(\alpha) = -j$. By Theorem 2, $0 = S(0) = a - 2aj$, thus $aS_0(\xi) = 2ajS_0(\xi) = 2aS_0(\xi)$, hence $aS_0(\xi) = 0$ for $\xi \in R$. Therefore $S_e(\xi) = a - aj = a/2$ for $\xi \in R$. However, $S_e(0) = 0$ by assumption, thus $S_e(\xi) = 0$ for $\xi \in R$, and $S = S_0 \in V_0(\alpha)$.

Corollary 2. $S \in V(0)$ implies $S(\xi) = a$ for $\xi \in R$, where $a^2 = 0$. Conversely, if $S(\xi) = a$ for $\xi \in R$ with $a^2 = 0$, then $S \in V(0)$.

PROOF. If $S \in V(0)$, then Theorem 2 and (4) imply $S_0(\xi) = -S_0(\xi)$, $S_0(\xi) = 0$ for $\xi \in R$. Hence $j = 0$, and $S(\xi) = a$ with $a^2 = 0$. The converse is trivial.

In view of this corollary we shall always assume that $\alpha \neq 0$.

Example. Let X be a Banach space, $B = B(X)$ the Banach algebra of the bounded linear operators in X , $J \in B$ an idempotent (projection) operator, $J \neq 0$, $J \neq I$. Then there exists an $A \in B$ such that $A^2 = 0$, $JA = 0$, $AJ \neq 0$.

Indeed, denoting the range of $T \in B$ by $R(T)$, we have $X = R(J) \oplus R(I - J)$, and there exist elements $u \in R(I - J)$, $u \neq 0$ and $u^* \in X^*$ such that $u^*(u) = 0$, and $u^*\{R(J)\} \neq 0$. For $x \in X$ set $Ax = u^*(x) \cdot u$, then $A \in B$, $R(A) \subset R(I - J)$, thus $JA = 0$. On the other hand, $AJx = u^*(Jx) \cdot u \neq 0$, and $A^2x = A\{u^*(x) \cdot u\} = u^*(x) \cdot u^*(u) \cdot u = 0$ for every $x \in X$, thus A fulfils the stated requirements.

Definition. We shall say that $S \in V(\alpha, B, j, a)$ if $S \in V(\alpha)$, B is a complex Banach algebra, $S_0(-\alpha) = j$, $S_e(2\alpha) = a$ and S is strongly measurable. We shall omit any of the variables of V if it is not emphasized.

Theorem 3. If $S \in V(\alpha, B, j, \cdot)$, then there exist one and only one $g_0 \in jBj$ and exactly one $a \in B$ such that $S(\xi) = \sin(\xi g_0) - a \cos(\xi g_0) + a - aj$, where by definition

$$\sin(\xi g_0) = \sum_{n=0}^{\infty} (-1)^n \frac{(\xi g_0)^{2n+1}}{(2n+1)!}, \quad \cos(\xi g_0) = j + \sum_{n=1}^{\infty} (-1)^n \frac{(\xi g_0)^{2n}}{(2n)!}.$$

Here $a = S_e(2\alpha)$, $a^2 = 0$, $ja = 0$, S is strongly differentiable at each point $\xi \in R$ and $S'(\xi) = g_0 \cos(\xi g_0) + ag_0 \sin(\xi g_0)$. Moreover, $S'(0) = S'_0(0) = g_0$.

PROOF. Under the conditions of the theorem $S_0 \in V(\alpha, B, j, \cdot)$ and, by Theorem 1, the function E associated with S_0 is also strongly measurable, and $E(0) = j$. Then by [6], 9.4. there exists exactly one $g \in jBj$ such that

$$E(\xi) = j + \sum_{n=1}^{\infty} \frac{(\xi g)^n}{n!} = \exp(\xi g),$$

and this series converges absolutely for $\xi \in R$. Writing $g_0 = -ig \in jBj$, we obtain that $S_0(\xi) = \sin(\xi g_0)$, and because of $S'_0(0) = g_0$ the uniqueness of g_0 is proved. Taking into account (4) and Theorem 1, we get $S_0(\xi + \alpha) = -S_0(\xi - \alpha) = -C(\xi)$, where C is the function associated with S_0 according to Theorem 1. But then

$$S_0(\xi + \alpha) = -\frac{1}{2} \{E(\xi) + E(-\xi)\} = -\frac{1}{2} \{\exp(i\xi g_0) + \exp(-i\xi g_0)\} = -\cos(\xi g_0),$$

and the remaining assertions are contained in Theorem 2.

Corollary (cf. [7], [2]). If B is a complex Banach-algebra, $C: R \rightarrow B$ is a strongly measurable function satisfying $C(\xi + \eta) + C(\xi - \eta) = 2C(\xi)C(\eta)$, and if for some $\alpha \in R$ ($\alpha \neq 0$) $C(\alpha) = 0$ holds, then $C(0) = j$ is idempotent and there exists a (not unique) $b \in jBj$ such that $C(\xi) = \cos(\xi b) = j + \sum_{n=1}^{\infty} (-1)^n \frac{(\xi b)^{2n}}{(2n)!}$.

PROOF. From the assumptions we obtain that $C(\xi + \alpha) = -C(\xi - \alpha)$ for $\xi \in R$. Putting the pair $(\xi + \alpha, \eta + \alpha)$ in d'Alembert's equation, the preceding relation gives

$$(14) \quad C(\xi + \eta + 2\alpha) - C(\xi - \eta + 2\alpha) = 2C(\xi + \alpha)C(\eta + \alpha).$$

If we set $S(\xi) = C(\xi + \alpha)$, then $S(0) = 0$ and (14) imply that $S \in V_0(\alpha)$, and clearly, S is strongly measurable. Thus, as in Theorem 3, for the function E associated with S we get $E(\xi) = j + \sum_{n=1}^{\infty} \frac{(\xi b)^n}{n!}$, where $j = C(0) = S(-\alpha)$ is idempotent and $b \in jBj$. From this it follows that $C(\xi) = 1/2 \{E(\xi) + E(-\xi)\} = \cos(\xi b)$, while $C(\xi) = \cos\{\xi(-b)\}$ is a trivial different representation.

Definition. If $S \in V(\alpha, B, \cdot, \cdot)$, then $\frac{dS}{d\xi}(0)$ will be called the generator element of S .

A characterization of the generator elements is given in the following

Theorem 4. Let B be a complex Banach algebra, $j \in B$ idempotent, $\alpha \in R$. $g_0 \in jBj$ is the generator element of some $S \in V(\alpha, B, j, \cdot)$ if and only if $g_0 = \sum_{\beta=1}^n \lambda_{\beta} j_{\beta}$, where $\lambda_{\beta} = \frac{\pi}{2\alpha} (4k_{\beta} - 1)$ (k_{β} are different integers for $\beta = 1, 2, \dots, n$), $j_{\beta} \in jBj$ are idempotents,

$$\sum_{\beta=1}^n j_{\beta} = j, \quad j_{\beta} j_{\gamma} = \delta_{\beta\gamma} j_{\beta}$$

($\gamma = 1, 2, \dots, n$). In this case we have

$$S_0(\xi) = \sum_{\beta=1}^n j_{\beta} \cdot \sin(\lambda_{\beta} \cdot \xi), \quad S_0(\xi + \alpha) = - \sum_{\beta=1}^n j_{\beta} \cdot \cos(\lambda_{\beta} \cdot \xi).$$

PROOF. According to the preceding theorems $g_0 \in jBj$ is the generator element of some $S \in V(\alpha; B, j, \cdot)$ if and only if $E(\xi) = S_0(\xi - \alpha) + iS_0(\xi) = \exp(i\xi g_0)$, where $\exp$ denotes the exponential function in jBj , and $E(\alpha) = \exp(i\alpha g_0) = -ij$. The latter is the case exactly when $i\alpha g_0 = \log(-ij)$, where $\log$ denotes logarithm in jBj . A value of this is clearly $\log^*(-ij) = -i\frac{\pi}{2}j$, and the spectrum of $-i\frac{\pi}{2} \cdot j \in jBj$ is $\{-i\frac{\pi}{2}\}$, which is incongruent (mod $2\pi i$) (see [5], p. 54.). Hence every logarithm of $-ij$ is of the following form (see loc. cit.): $\log(-i \cdot j) = -i\frac{\pi}{2} \cdot j + i2\pi \sum_{\beta=1}^n k_\beta \cdot j_\beta$, where k_β ($\beta = 1, 2, \dots, n$) are different integers, $j_\beta \in jBj$, $j_\beta \cdot j_\gamma = \delta_{\beta\gamma} \cdot j_\beta$ ($\beta, \gamma = 1, 2, \dots, n$), $\sum_{\beta=1}^n j_\beta = j$. Therefore we obtain that

$$g_0 = \frac{\pi}{2\alpha} \left\{ -\sum_{\beta=1}^n j_\beta + 4 \sum_{\beta=1}^n k_\beta j_\beta \right\} = \frac{\pi}{2\alpha} \sum_{\beta=1}^n (4k_\beta - 1) j_\beta.$$

Because of the properties of the elements j, j_β we have in this case

$$\begin{aligned} E(\xi) &= \exp \left(i\xi \sum_{\beta=1}^n \lambda_\beta j_\beta \right) = \prod_{\beta=1}^n \exp(i \cdot \xi \cdot \lambda_\beta \cdot j_\beta) = \prod_{\beta=1}^n \{j + j_\beta \cdot [\exp(i\xi \lambda_\beta) - 1]\} = \\ &= j + \sum_{\beta=1}^n j_\beta [\exp(i\xi \lambda_\beta) - 1] = \sum_{\beta=1}^n j_\beta \exp(i\xi \lambda_\beta), \end{aligned}$$

and hence $S_0(\xi) = \sum_{\beta=1}^n j_\beta \sin(\lambda_\beta \cdot \xi)$. For the function C associated with S_0 $S_0(\xi + \alpha) = -S_0(\xi - \alpha) = -C(\xi)$ holds and thus the remaining statement is also proved.

2. The equation in the algebra $B(X)$.

Let X be a complex Banach space and $B(X)$ the algebra of the bounded linear operators in X .

Definition. We say that $S \in V(\alpha, B(X), J, A)$ if $S: R \rightarrow B(X)$ satisfies (1), $S_0(-\alpha) = J$ is a projection operator, $S_e(2\alpha) = A$ and S is a strongly measurable operator function. Moreover, we say that $S \in V_0(\alpha, B(X), J)$ if, in addition, S is odd (then necessarily $A = 0$).

Clearly, if $S \in V(\alpha, B(X), J, A)$, then S_0 and the function E associated with S_0 are also strongly measurable, hence E and S are continuous in the strong operator topology of $B(X)$.

Definition. If $S \in V(\alpha, B(X), J, A)$, then we set $S'(\xi)x = \lim_{\eta \rightarrow 0} \frac{S(\xi + \eta) - S(\xi)}{\eta} x$ if this limit exists in the strong topology of X , and then we write $x \in D\{S'(\xi)\}$. We denote $S'(0)$ by $G_0 = G_0(S)$, and call it the generator operator of S .

Lemma 1. If $S \in V_0(\alpha, B(X), J)$ and $x \in D(G_0)$, then for every $\xi \in R$ we have $x \in D\{S'(\xi)\}$, $S(\xi)x \in D(G_0)$ and

$$S'(\xi)x = S(\xi - \alpha)G_0x = G_0S(\xi - \alpha)x.$$

PROOF. By (1) and the strong continuity of S , we obtain for $x \in D(G_0)$, $\xi \in R$ that there exists

$$(15) \quad S'(\xi)x = \lim_{\eta \rightarrow 0} \frac{S(\xi + \eta) - S(\xi)}{\eta} x = \lim_{\eta \rightarrow 0} \frac{S(\xi - \alpha + \eta/2)S(\eta/2)}{\eta/2} x = S(\xi - \alpha)G_0x.$$

On the other hand, for $x \in D(G_0)$

$$\begin{aligned} G_0S(\xi - \alpha)x &= \lim_{\eta \rightarrow 0} \frac{1}{\eta} S(\eta)S(\xi - \alpha)x = \lim_{\eta \rightarrow 0} \frac{S(\xi + \eta) - S(\xi - \eta)}{2\eta} x = \\ &= \lim_{\eta \rightarrow 0} \frac{1}{2} \cdot \frac{S(\xi + \eta) - S(\xi) - \{S(\xi - \eta) - S(\xi)\}}{\eta} x = S'(\xi)x \end{aligned}$$

exists because of (15), and the proof is complete.

As a corollary we get that $x \in D(G_0)$ implies $G_0x = S(-\alpha)G_0x$, hence $R(G_0) \subset R\{S(-\alpha)\} = R(J)$.

Lemma 2. If $S \in V(\alpha, B(X), J, A)$ and S_0 is its odd part, then $G_0(S) = G_0(S_0)$.

PROOF. If $x \in D\{G_0(S)\}$, then

$$\begin{aligned} G_0(S_0)x &= \lim_{\xi \rightarrow 0} \frac{S_0(\xi)}{\xi} x = \lim_{\xi \rightarrow 0} \frac{S(\xi) - S(-\xi)}{2\xi} x = \\ &= \lim_{\xi \rightarrow 0} \frac{S(\xi) - S(0) - \{S(-\xi) - S(0)\}}{2\xi} x \end{aligned}$$

also exists, and $G_0(S_0)x = G_0(S)x$. Conversely, if $x \in D\{G_0(S_0)\}$, then by Lemma 1 $x \in D\{S'_0(\alpha)\}$ and $S'_0(\alpha)x = 0$. Applying Theorem 2, we obtain that

$$\lim_{\xi \rightarrow 0} \frac{S_e(\xi) - S_e(0)}{\xi} x = A \lim_{\xi \rightarrow 0} \frac{S_0(\xi + \alpha) - S_0(\alpha)}{\xi} x = 0,$$

and thus exists

$$G_0(S)x = \lim_{\xi \rightarrow 0} \frac{S(\xi) - S(0)}{\xi} x = \lim_{\xi \rightarrow 0} \frac{S_0(\xi)}{\xi} x = G_0(S_0)x,$$

which ends the proof.

Suppose $S \in V(\alpha, B(X), J, A)$, S_0 is its odd part and C, E are the functions associated with S_0 as in Theorem 1. Introduce the following notations: $G = E'(0)$, $G_1 = C'(0)$ and $G_2 = C''(0)$, where the domains of the right sides consist of exactly those $x \in X$, for which the respective derivatives exist in the strong topology of X . We define the operators $C'(\xi)$ ($\xi \in R$) in a similar manner.

Lemma 3. If $S \in V(\alpha, B(X), J, A)$, then for every $\xi \in R$ we have $D(G_0) \subset D\{C'(\xi)\}$, and $x \in D(G_1)$ implies $G_1x = 0$. Moreover, $G = iG_0$ and $G_2 = -G_0^2$.

PROOF. By Lemmas 1 and 2, $x \in D(G_0)$ implies $C'(\xi)x = S'_0(\xi - \alpha)x = -S_0(\xi)G_0x$ for $\xi \in R$. Moreover, $G_1 = C'(0) = S'_0(-\alpha)$ and, in view of (3) and (4), $S_0(\eta - \alpha) = S_0(-\eta - \alpha)$. Hence for $x \in D(G_1)$ we get

$$\begin{aligned} G_1x &= \lim_{\eta \rightarrow 0} \frac{S_0(-\alpha + \eta) - S_0(-\alpha)}{\eta} x = \\ &= \lim_{\eta \rightarrow 0} \frac{1/2 \{S_0(-\alpha + \eta) + S_0(-\alpha - \eta)\} - S_0(-\alpha)}{\eta} x = 0. \end{aligned}$$

If $x \in D(G_0)$, then $x \in D(G_1)$ and exists

$$Gx = \lim_{\eta \rightarrow 0} \left\{ \frac{C(\eta) - C(0)}{\eta} x + i \frac{S_0(\eta)}{\eta} x \right\} = iG_0x,$$

while $x \in D(G)$ implies

$$G_0x = \lim_{\eta \rightarrow 0} \frac{-i}{2} \left\{ \frac{E(\eta) - E(0)}{\eta} x + \frac{E(-\eta) - E(0)}{-\eta} x \right\} = \frac{1}{i} Gx,$$

hence $G = iG_0$.

To prove the last assertion suppose $x \in D(G_0)$, $\eta \neq 0$. Then

$$\frac{C'(\eta) - G_1}{\eta} x = -\frac{S_0(\eta)G_0x}{\eta}, \quad \text{thus} \quad D(G_0^2) \subset D(G_2),$$

and $x \in D(G_0^2)$ implies $G_2x = -G_0^2x$. On the other hand, putting $X_1 = JX$, $X_0 = (I - J)X$, X is the direct sum of X_1 and X_0 . From Theorem 1 it follows that G_0, G and G_2 are identically 0 on X_0 , while on X_1 $E(\xi)$ is a strongly continuous group of operators and $C(\xi)$ a strongly continuous cosine operator function (cf. [10]) with $C(0) = J$, the identical operator in X_1 . If $\bar{G}_0, \bar{G}, \bar{G}_2$ denote the restrictions of the respective operators to X_1 , then there exists a $v > 0$ such that $z > v$ implies $z^2 \in \rho(\bar{G}_2)$, $z \in \rho(\bar{G}) \cap \rho(-\bar{G})$, where ρ denotes the resolvent set. Hence also $z^2 \in \rho(\bar{G}^2)$, thus

$$(z^2J - \bar{G}^2)D(\bar{G}^2) = X_1 = (z^2J - \bar{G}_2)D(\bar{G}_2)$$

the operators on both sides being one-to-one mappings. Therefore we get $D(\bar{G}_2) = D(\bar{G}^2)$, whence $D(G_0^2) = D(G^2) = D(G_2)$ and $G_2 = -G_0^2$.

Remark. In the proof we employed an idea of M. SOVA [10].

Corollary. $D(G_0)$ is dense in X and G_0 is a closed operator. $D(G_0) = X$, i.e. G_0 is bounded if and only if $S(\xi)$ is continuous in the uniform operator topology.

Lemma 4. If $S \in V_0(\alpha, B(X), J)$, $y \in X$ and $A, B \in R$, then $x_{A,B}(y) = \int_A^B S(\eta)y d\eta$ belongs to $D(G_0)$ and $G_0x_{A,B}(y) = \{S(B + \alpha) - S(A + \alpha)\}y$. Moreover, $R(J) = R(G_0)$.

PROOF. If E is the group of operators associated with S , then we have

$$\int_A^B S(\eta) y d\eta = \frac{1}{2i} \int_A^B \{E(\eta) - E(-\eta)\} y d\eta = \frac{1}{2i} \left\{ \int_A^B E(\eta) y d\eta + \int_{-A}^{-B} E(\eta) y d\eta \right\}.$$

But then, according to a well-known result of N. DUNFORD, there exists

$$\begin{aligned} G_0 x_{A,B}(y) &= -iGx_{A,B}(y) = -\frac{1}{2} \{E(B) - E(A) + E(-B) - E(-A)\} y = \\ &= -\{C(B) - C(A)\} y = \{S(B+\alpha) - S(A+\alpha)\} y \end{aligned}$$

where C is the function associated with S , occurring in Theorem 1. Hence with $B = -2\alpha$, $A = -\alpha$ for every $y \in X$ we get $S(-\alpha)y = G_0 x_{-\alpha, -2\alpha}(y)$, $R(J) = R\{S(-\alpha)\} \subset R(G_0)$. The converse inclusion was established after Lemma 1.

Corollary. $S \in V_0(\alpha, B(X), J)$ is continuous in the uniform operator topology if and only if $D(G_0^2) = D(G_0)$.

PROOF. If $D(G_0^2) = D(G_0)$, then with the notation of Lemma 3, $X_1 = R(G_0) \subset D(G_0)$, thus $D(G_0) = X$ and G_0 is bounded. The converse is evident.

Remark 1. Suppose $S \in V_0(\alpha, B(X), J)$. It can be seen from the proof of Lemma 3 that there is no essential restriction in assuming that $J = S(-\alpha) = I$. To simplify the statements, this assumption will often be made in what follows.

Remark 2. $S \in V(\alpha, B(X), J, A)$ and $S(-\alpha) = I$ hold if and only if $S \in V_0(\alpha, B(X), I)$. Indeed, from Theorem 2 we see that $I = S_0(-\alpha) + S_e(-\alpha) = J + A - AJ$ implies $0 = (A - AJ)^2 = I - J$, hence $J = I$ and $A = JA = 0$. Consequently, we have $S \in V_0(\alpha, B(X), I)$, while the converse is evident.

Theorem 5. If $S \in V_0(\alpha, B(X), I)$ and γ_n denotes $\frac{\pi}{2\alpha}(4n-1)$ where n is integer, then the spectrum of G_0 is entirely point spectrum, and $\mathbb{Q} \neq \sigma(G_0) \subset \{\gamma_n; n=0, \pm 1, \pm 2, \dots\}$. Moreover, there exist projection operators P_n ($n=0, \pm 1, \pm 2, \dots$) in X , for which $P_n X = \{x \in D(G_0); G_0 x = \gamma_n x\} \neq \{0\}$ if and only if γ_n is an eigenvalue of G_0 . $P_n P_k = \delta_{nk} P_n$, and for $x \in X$ $S(\xi)x = (C, 1) - \sum'_{n=-\infty}^{\infty} \sin(\gamma_n \xi) P_n x$, $x = (C, 1) - \sum'_{n=-\infty}^{\infty} P_n x$. For $x \in D(G_0)$ we have $G_0 x = (C, 1) - \sum'_{n=-\infty}^{\infty} \gamma_n P_n x$ (here $(C, 1) - \sum'$ denotes Cesàro sum of the first order, and $\sum'$ indicates that summation is extended only for those n , for which $\gamma_n \in \sigma(G_0)$). S is continuous in the uniform operator topology if and only if the number of the eigenvalues of G_0 is finite, and then

$$S(\xi) = \sum'_n \sin(\gamma_n \xi) \cdot P_n, \quad G_0 = \sum'_n \gamma_n \cdot P_n, \quad I = \sum'_n P_n.$$

PROOF. If E is the group of operators associated with S , then $E(\alpha) = -iI$, and the eigenvalue $-i$ is the unique element of $\sigma\{E(\alpha)\}$. If G is the generator operator of E , then according to the theorems of [6], 16.7. $\mathbb{Q} \neq P\sigma(G) \subset \sigma(G) \subset$

$\subset \{\beta_n; n=0, \pm 1, \pm 2, \dots\}$, where $\beta_n = i\gamma_n$, and thus Lemma 3 implies the statements concerning $\sigma(G_0)$ except $P_\sigma(G_0) = \sigma(G_0)$. However, this will follow from Theorem 6, which implies that all the singularities of the resolvent of G_0 are poles, hence eigenvalues of G_0 .

Modifying the ideas in the proof of [6], Theorem 16.7.2., we get that the operators $U(\tau) = e^{i\frac{\pi}{2\alpha}\tau} E(\tau)$ ($\tau \in \mathbb{R}$) in X form a strongly continuous group with period $|\alpha|$. For $x \in X, n=0, \pm 1, \pm 2, \dots$ we define $P_n x = \frac{1}{\alpha} \int_0^\alpha e^{-in\frac{2\pi}{\alpha}\tau} U(\tau)x d\tau$. Then $P_n \in B(X)$, $P_n P_k = \delta_{nk} P_n$, and $P_n X = \{x \in D(G): Gx = \beta_n x\} = \{x \in D(G_0): G_0 x = \gamma_n x\}$. $P_n X \neq \{0\}$ if and only if γ_n is an eigenvalue of G_0 , and the strong continuity of $U(\xi)x$ implies for $x \in X$

$$U(\xi)x = (C, 1) - \sum'_{n=-\infty}^{\infty} e^{in\frac{2\pi}{\alpha}\xi} P_n x,$$

and thus

$$(16) \quad E(\xi)x = (C, 1) - \sum'_{n=-\infty}^{\infty} e^{i\gamma_n \xi} P_n x,$$

hence $S(\xi)x = (C, 1) - \sum'_{n=-\infty}^{\infty} \sin(\gamma_n \xi) P_n x$. (16) with $\xi = 0$ gives

$$(17) \quad x = (C, 1) - \sum'_{n=-\infty}^{\infty} P_n x.$$

P_n and $U(\xi)$ commute, hence so do P_n and $S(\xi)$. Therefore $x \in D(G_0)$ implies $P_n x \in D(G_0)$ and $P_n G_0 x = G_0 P_n x = \gamma_n P_n x$. Thus, by (17), we have for $x \in D(G_0)$

$$G_0 x = (C, 1) - \sum'_{n=-\infty}^{\infty} P_n G_0 x = (C, 1) - \sum'_{n=-\infty}^{\infty} \gamma_n P_n x.$$

If S is continuous in the uniform operator topology, then the spectral radius of G_0 , $r(G_0) \leq \|G_0\|$, hence the number of the eigenvalues of G_0 is finite. Conversely, if $P\sigma(G_0)$ is finite, then $S(\xi)x$ reduces to a finite sum, and for every $x \in X$ we have $x = \sum' P_n x$, $S(\xi)x = \sum' \sin(\gamma_n \xi) P_n x$, and

$$G_0 x = \lim_{\xi \rightarrow 0} \frac{S(\xi)x}{\xi} = \lim_{\xi \rightarrow 0} \sum' \frac{\sin(\gamma_n \xi)}{\xi} P_n x = \sum' \gamma_n P_n x,$$

hence $D(G_0) = X$ and S is continuous in the uniform operator topology. Thus the proof is complete.

A characterization of the generator operator G_0 is given in the following

Theorem 6. Let G_0 be a closed operator in X with dense domain $D(G_0)$. G_0 is the generator of some $S \in V_0(\alpha, B(X), I)$ ($\alpha \neq 0$) if and only if there exists a real number $M \geq 1$ such that 1° and 2° hold:

1° $(I+zG_0)^{-1}$ is a bounded operator and $\|(I+zG_0)^{-n}\| \leq M$ for every purely imaginary z and for $n=1, 2, \dots$

2° the function $F(z) = (e^{-iz} - i)R(z; G_0)$ $\{z \in \varrho(G_0)\}$ has an analytical continuation $H(z)$ on the whole complex plane for which $\|H(z)\| \leq Ke^{|z \operatorname{Im} z|}$ (here R denotes the resolvent operator, $K > 0$).

In this case the function $S(\xi)$ generated by G_0 is determined uniquely, and $\|S(\xi)\| \leq M$ for $\xi \in \mathbb{R}$.

Remark. A similar theorem has been proved in [9] (Teor. 2.). However, the proof of the sufficiency is not complete there and, in fact, can not dispense with a boundedness condition similar to that in 2°. In addition, our proof seems to be simpler.

PROOF. 1. Suppose $S \in V_0(\alpha, B(X), I)$, G_0 is its generator, E is the group of operators associated with S and $G = iG_0$.

Then 1° follows from Lemma 3 and [6], Theor. 12.3.2. Put $z_k = \frac{i\pi}{2\alpha}(4k-1)$ (k integer), and for $z \neq z_k$, $x \in X$ $U(z)x = (1 + ie^{-\alpha z})^{-1} \int_0^{\alpha} e^{-\xi z} E(\xi)x d\xi$. From $E(\alpha) = -iI$ it follows for $x \in D(G)$ that $U(z)Gx = zU(z)x - x$, and by the denseness of $D(G)$ that $R(z; G) = U(z)$.

For $z \in \varrho(G_0)$ we get $R(z; G_0) = iR(iz; G)$, by Lemma 3. Hence $z \neq -iz_k$, $x \in X$ implies $(e^{-iz} - i)R(z; G_0)x = (1 + ie^{-iz})R(iz; G)x = \int_0^{\alpha} e^{-i\xi z} E(\xi)x d\xi$. Putting $H(z)x = \int_0^{\alpha} e^{-i\xi z} E(\xi)x d\xi$ for every complex z , $H(z)$ is analytic and a continuation of $F(z)$, for $R(z; G_0)$ is holomorphic in $\varrho(G_0)$. Moreover, we have

$$\|H(z)\| \leq e^{|z \operatorname{Im} z|} \int_0^{\alpha} \|E(\xi)\| d\xi = Ke^{|z \operatorname{Im} z|},$$

and 2° is also verified.

2. If G_0 satisfies 1°, then $G = iG_0$ generates the strongly continuous group of operators E , for which $\|E(\xi)\| \leq M$ ($\xi \in \mathbb{R}$). We show that 2° even implies $E(\xi + \alpha) = -iE(\xi)$ for $\xi \in \mathbb{R}$.

If $Q(z) = H(-iz)$, then $\|Q(z)\| \leq Ke^{|z \operatorname{Re} z|}$, and for $\operatorname{Re} z > 0$, $x \in X$ we get

$$\frac{Q(z)x}{1 + ie^{-\alpha z}} = R(z; G)x = \int_0^{\infty} e^{-\xi z} E(\xi)x d\xi.$$

Suppose $u, v > 0$ and $E_2(u)x = \int_0^u \int_0^v E(\xi)x d\xi dv$, then [6], (6.3.9) gives for $r > 0$

$$E_2(u)x = \lim_{B \rightarrow \infty} \frac{1}{2\pi i} \int_{r-iB}^{r+iB} e^{zu} \frac{Q(z)x}{1 + ie^{-\alpha z}} \cdot z^{-2} dz.$$

In the integrand $(1+ie^{-az})^{-1}$ is bounded, and the absolute convergence of the integral yields with the notation $g(z)=(2\pi i)^{-1}e^{zu}(1+ie^{-az})^{-1}z^{-2}Q(z)x$ that $E_2(u)x = \int_{r-i\infty}^{r+i\infty} g(z)dz$. Making use of 2° , it can be shown after some calculation that $E_2(u)x = 2\pi i \left\{ \text{Res}(g; 0) + \sum_{p=-\infty}^{\infty} \text{Res}(g; z_p) \right\}$ for $u > |\alpha|$. With the notation

$$Q^* = \left\{ \frac{d}{dz} [(1+ie^{-az})^{-1}Q(z)x] \right\}_{z=0}$$

we get for $u > |\alpha|$

$$E_2(u)x = Q^* + u(1+i)^{-1}Q(0)x + \sum_{p=-\infty}^{\infty} e^{uz_p}(\alpha z_p^2)^{-1}Q(z_p)x.$$

Thus for $u > 2|\alpha|$ we obtain

$$iE_2(u+\alpha)x - E_2(u)x = (i-1)Q^* + \left(u \cdot \frac{i-1}{i+1} + \frac{i\alpha}{1+i} \right) Q(0)x$$

and, differentiating twice, $E(u+\alpha)x = -iE(u)x$. If $\xi \in R$, then applying $E(\xi-u)$ to both sides we get

$$E(\xi+\alpha)x = -iE(\xi)x \quad (x \in X).$$

If we put $S(\xi) = -\frac{i}{2} \{E(\xi) - E(-\xi)\}$, then $S \in V_0(\alpha, B(X), I)$ by Theorem 1, and Lemma 3 gives that the generator of S is G_0 . The uniqueness of S follows from the fact that G determines the group E uniquely. $\|S(\xi)\| \leq M$ obtains immediately, and the proof is complete.

Since the generator G_0 uniquely determines the function $S \in V_0(\alpha, B(X), I)$, this can be denoted by $S(\xi; G_0)$ if we wish to emphasize the generator.

Theorem 7. If $S(\xi; G_0) \in V_0(\alpha, B(X), I)$ and $v > 0$, then $S(\xi; vG_0) \in V_0\left(\frac{\alpha}{v}, B(X), I\right)$, and for every $x \in X$ we have $\lim_{v \rightarrow 1} S(\xi; vG_0)x = S(\xi; G_0)x$.

PROOF. The first half of the theorem is clearly true by Theorem 6. If $E(\xi; iA)$ is the group of operators associated with $S(\xi; A)$, then according to the proof of [6], Theorem 12.3.1., for $x \in X$ we obtain $E(\xi; ivG_0)x = \lim_{n \rightarrow \infty} \left(I - \frac{\xi}{n} ivG_0 \right)^{-n} x = \lim_{n \rightarrow \infty} \left(I - \frac{\xi \cdot v}{n} iG_0 \right)^{-n} x = E(\xi \cdot v; iG_0)x$. Since $E(\xi; iG_0)$ is strongly continuous, for $x \in X$ we have $\lim_{v \rightarrow 1} E(\xi; ivG_0)x = E(\xi; iG_0)x$, and the second half of the theorem follows by Theorem 1.

Remark. Let X be the complex plane, $g_0 \in X$, $S(\xi)\xi = \sin(g_0\xi) \cdot \xi$ for $\xi \in R$, $\xi \in X$. If $S(\xi; g_0) \in \bigcup_{\alpha \in R \setminus \{0\}} V_0(\alpha, B(X), I)$, then S is periodic, consequently g_0 is real. Therefore we can not expect that the admissible perturbations of G_0 are much more general than those in Theorem 7.

Theorem 8. Let H be a Hilbert space and $S \in V_0(\alpha, B(H), I)$ with generator G_0 . Then there exists a bounded positive selfadjoint operator Q such that $B = QG_0Q^{-1}$ is a (generally nonbounded) selfadjoint operator with $D(B) = Q\{D(G_0)\}$, and $S(\xi) = Q^{-1} \sin(\xi \cdot B)Q$. Moreover, with the notations of Theorem 5 we have for $x \in H$

$$S(\xi)x = \sum'_{n=-\infty}^{\infty} \sin(\xi \gamma_n) \cdot P_n x,$$

for $x \in D(G_0)$

$G_0 x = \sum'_{n=-\infty}^{\infty} \gamma_n \cdot P_n x$, and the projections $R_n = QP_nQ^{-1}$ are selfadjoint.

PROOF. If E is the group associated with S , then $\sup_{\xi \in R} \|E(\xi)\| \leq K$ implies,

according to [8], that there exists a bounded selfadjoint operator Q with $\frac{1}{K} \cdot I \leq Q \leq K \cdot I$ such that $QE(\xi)Q^{-1}$ is unitary for $\xi \in R$. Then there exists a selfadjoint B such that $E(\xi) = Q^{-1}e^{i\xi B}Q$, and thus $S(\xi) = Q^{-1} \sin(\xi B)Q$, where clearly $\sin(\xi B) \in V_0(\alpha, B(H), I)$ and B is its generator. If $Qx \in D(B)$, then there exists $G_0 x = \lim_{\xi \rightarrow 0} \frac{1}{\xi} Q^{-1} \sin(\xi B)Qx = Q^{-1}BQx$, thus $Q^{-1}\{D(B)\} \subset D(G_0)$. If $Q^{-1}x \in D(G_0)$, then $Bx = \lim_{\xi \rightarrow 0} \frac{1}{\xi} \sin(\xi B)x = \lim_{\xi \rightarrow 0} \frac{1}{\xi} QS(\xi)Q^{-1}x = QG_0Q^{-1}x$, thus $Q\{D(G_0)\} \subset D(B)$, hence $D(B) = Q\{D(G_0)\}$ and $B = QG_0Q^{-1}$ as stated.

B and G_0 are similar, hence $\sigma(B) = \sigma(G_0)$ and $P\sigma(B) = P\sigma(G_0)$ (cf. [4], Problem 60.). The spectral representation of B gives for $x \in D(B)$ that $Bx = \sum'_{n=-\infty}^{\infty} \gamma_n \cdot R_n x$, where R_n is a selfadjoint projection ($n=0, \pm 1, \pm 2, \dots$) with $R_n R_k = \delta_{nk} R_n$. From this we obtain for $x \in D(G_0)$ that $G_0 x = \sum'_{n=-\infty}^{\infty} \gamma_n \cdot Q^{-1} R_n Qx$,

and for $x \in H$ that $S(\xi)x = \sum'_{n=-\infty}^{\infty} \sin(\xi \gamma_n) Q^{-1} R_n Qx$. Denote $Q^{-1} R_n Q$ by Q_n , we shall show that $Q_n = P_n$.

If $x \in Q_n H$, then $S(\xi)x = \sin(\xi \cdot \gamma_n)x$ and $G_0 x = \gamma_n x$, thus, by Theorem 5, $x \in P_n H$. Conversely, $x \in P_n H$ implies $iG_0 x = i\gamma_n \cdot x$, hence $E(\xi)x = e^{i\gamma_n \xi} x$ and

$$\sin(\gamma_n \xi) \cdot x = S(\xi)x = \sum'_{k=-\infty}^{\infty} \sin(\xi \cdot \gamma_k) Q_k x.$$

Applying P_n to both sides, we get because of $Q_k H \subset P_k H$ that $\sin(\gamma_n \xi)x = \sin(\gamma_n \xi) Q_n x$, and hence $x \in Q_n H$. Thus $P_n = Q^{-1} R_n Q$ for every integer n and the proof is complete.

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