

On polynomial regression of polynomial and linear statistics

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1. Introduction

In [3], YU. V. LINNIK and A. A. ZINGER proved the following theorem:

Let $\xi_1, \dots, \xi_n$ be a sample from random variable ξ and assume that ξ has moments up to order m . Let the adjoint polynomial of the polynomial statistic

$$P = \sum A_{v_1 \dots v_n} \xi_1^{v_1} \dots \xi_n^{v_n}$$

be a regular polynomial statistic of degree p_1 and order $m |m \leq p_1|$. Here and in the sequel the summation is extended over all nonnegative integers $v_1, \dots, v_n$ which satisfy the relation

$$(1.1) \quad v_1 + \dots + v_n \leq p_1.$$

If P has constant regression on $A = \xi_1 + \dots + \xi_n$ and $m \geq n-1$ then the characteristic function $f(t)$ of ξ is an entire function.

In [2], B. GYIRES generalized this theorem in two directions. First he supposed only that $\xi_1, \dots, \xi_n$ are independent random variables instead of being a sample, consisting of independently and identically distributed random variables. Secondly he avoided the condition $m \geq n-1$. It is well known that this condition strongly narrows down the applicability of the theorem of Linnik and Zinger.

In this paper we also avoid the condition $m \geq n-1$. We suppose that $\xi_1, \dots, \xi_n$ are independent but not necessarily identically distributed random variables and that they have moments of all orders. We give an extension of the theorem of Linnik and Zinger in the case, when P has regression of order r ($0 \leq r \leq m-1$) on $A = \sum_{j=1}^n \alpha_j \xi_j$, $\alpha_j \neq 0$, $\alpha_j \in R_1$, $j=1, 2, \dots, n$. We shall use the following notations, definitions and results.

Let R_n be the n -dimensional space with row vectors as elements. If $v \in R_n$ then v^* stands for the transpose of the vector v .

$a \in R_n$ is the vector with all components equal to 1.

Let the components of the random vector-variable $\xi = (\xi_j) \in R_n$ be independent random variables. Denote the distribution function of ξ_j by $F_j(x)$ ($j=1, 2, \dots, n$) and assume that $F_j(x)$ has moments up to order m . Let

$$P = \sum A_{v_1 \dots v_n} \xi_1^{v_1} \dots \xi_n^{v_n}$$

be a polynomial statistic of degree p_1 .

Definition 1.1. The statistic P is said to be a common polynomial statistic of degree p_1 and order $m | m \leq p_1 |$ if the following conditions are satisfied:

- a) The statistic P is a nonnegative polynomial;
- b) No exponent in P exceeds m ;
- c) P contains the m -th power of each variable. (If the random variables $\xi_1, \dots, \xi_n$ are identically distributed then it is sufficient to assume that the polynomial P of degree p_1 and order $m \leq p_1$ contains the m -th power of at least one variable. In this case P is a regular polynomial statistic of degree p_1 and order m .)

Definition 1.2. A characteristic function $f(t)$ is said to be an analytic characteristic function if there exists a function $A(z)$ of the complex variable z which is regular in the circle $|Z| < \varrho$ ($\varrho > 0$) and a constant $\Delta > 0$ such that $A(t) = f(t)$ for $|t| < \Delta$.

Lemma 1.1. ([1], p. 89, lemma 5.3.4.) *If a characteristic function $f(z)$ is regular in a neighbourhood of the origin then it is also regular in a horizontal strip and it can be represented in this strip by the Fourier integral $f(z) = \int_{-\infty}^{\infty} e^{izx} dF(x)$ where $F(x)$ is the distribution function corresponding to $f(z)$. This strip is either the whole plane, or it has one, or two, horizontal boundary lines.*

Definition 1.3. If the strip is the whole plane then $f(z)$ is called an entire characteristic function.

Definition 1.4. Consider two random variables ξ and η and assume that the conditional expectation $E(\eta|\xi)$ exists. We say that η has polynomial regression of order r on ξ if the relation

$$E(\eta|\xi) = \beta_0 + \beta_1 \xi + \dots + \beta_r \xi^r$$

holds almost everywhere.

Assume that the first moments of η and the r -th moment of ξ exist. Then

$$E(\eta) = \beta_0 + \beta_1 E(\xi) + \dots + \beta_r E(\xi^r).$$

Theorem 1.1. ([1], p. 103, Theorem 6.1.1.) *Let ξ and η be two random variables and assume that the expectations $E(\eta)$ and $E(\xi^r)$ exist where r is a nonnegative integer. The random variable η has polynomial regression of order r on ξ if and only if the relation*

$$E(\eta e^{it\xi}) = \sum_{j=0}^r \beta_j E(\xi^j e^{it\xi})$$

holds for all real t . Here the β_j are real constants.

2. A new version of Linnik—Zinger's theorem

First we prove the following theorem.

Theorem 2.1. *Let the components of the random vector-variable $\zeta = (\zeta_j) \in R_n$ be independent but not necessarily identically distributed random variables. Let the distribution function of ζ_j be denoted by $F_j(x)$ and assume that ζ_j has moments of all orders, $j=1, 2, \dots, n$. Let $A = \zeta a^*$ and let*

$$P = \sum A_{v_1 \dots v_n} \zeta_1^{v_1} \dots \zeta_n^{v_n}$$

be a common polynomial statistic of degree p_1 and order m ($m \leq p_1$). If P has polynomial regression of order r ($0 \leq r \leq m-1$) on A then the characteristic function $f_j(t)$ of $F_j(x)$ is an entire characteristic function.

To prove theorem 2.1. the following lemma is needed.

Lemma 2.1. ([2], lemma 3.1.) *Let α be a positive constant. Let s and k be positive integers, $s \geq k$. Put*

$$S_{k\alpha}(s) = \sum \left(\frac{s}{j_1 \dots j_k} \right)^{1+\alpha}$$

where the summation is extended over all positive integers $j_1, \dots, j_k$ which satisfy the relation $j_1 + \dots + j_k = s$. Then

$$S_{k\alpha}(s) = (2^{2+\alpha} \cdot s_\alpha)^{k-1}$$

where s_α is the sum of the series

$$1 + \left(\frac{1}{2}\right)^{1+\alpha} + \left(\frac{1}{3}\right)^{1+\alpha} + \dots$$

PROOF of theorem 2.1.

Since P has polynomial regression of order r ($0 \leq r \leq m-1$) on A it is easily seen that the characteristic functions $f_j(t)$ of ζ_j ($j=1, 2, \dots, n$) satisfy the following equation:

$$(2.1) \quad \int_{R_n} P(x) e^{itxa^*} dF_1(x_1) \dots dF_n(x_n) = \sum_{p=0}^r \gamma_p \int_{R_n} (xa^*)^p e^{itxa^*} dF_1(x_1) \dots dF_n(x_n).$$

Since ζ_j ($j=1, 2, \dots, n$) has moments of all orders, we may differentiate (2.1) any number of times. By differentiating both sides of (2.1) N -times then putting $t=0$ we obtain

$$(2.2) \quad \int_{R_n} P(x) (xa^*)^N dF_1(x_1) \dots dF_n(x_n) = \sum_{p=0}^r \gamma_p \int_{R_n} (xa^*)^{N+p} dF_1(x_1) \dots dF_n(x_n).$$

Denote the k -th absolute moment of $F_j(x)$ by

$$(2.3) \quad \beta_k^{(j)} = \int_{-\infty}^{\infty} |x|^k dF_j(x_j) \quad |j=1, \dots, n|.$$

First we show that the inequality

$$(2.4) \quad \beta_k^{(j)} \equiv \frac{k!}{k^{1+\alpha}} M^k$$

holds for any positive integer k where α and M are positive constants which will be chosen later. We prove (2.4) by induction.

Since P is a common polynomial statistic it is nonnegative and so its order m is necessarily an even integer. Therefore we can write $|m \equiv 2|$

$$(2.5) \quad P = P_0(\xi_2, \dots, \xi_n) \xi_1^m + \dots + P_m(\xi_2, \dots, \xi_n)$$

where $P_0(\xi_2, \dots, \xi_n)$ is also a nonnegative polynomial in the $n-1$ variables $\xi_2, \dots, \xi_n$. First let N (and thus k) be an even integer. We use the inequalities

$$(2.6) \quad ||x_1| - |x_2 + \dots + x_n|| \leq |x_1^*| \leq |x_1| + \dots + |x_n|$$

and obtain from (2.2) that

$$(2.7) \quad \begin{aligned} \int_{R_n} P(x) (|x_1| - |x_2 + \dots + x_n|)^N dF_1(x_1) \dots dF_n(x_n) &\leq \\ &\leq \sum_{p=0}^r |\gamma_p| \int_{R_n} \left(\sum_{j=1}^n |x_j| \right)^{N+p} dF_1(x_1) \dots dF_n(x_n) \end{aligned}$$

Theorem 2.1. is trivial if the ξ_j 's are bounded random variables. So assume, from now on, that at least one of the random variables $\xi_1, \dots, \xi_n$ is not bounded. Let

$$(2.8) \quad 1 - F_1(x-0) + F_1(-x) > 0.$$

It is always possible to find a bounded region $\Omega \subset R_{n-1}$ such that

$$(2.9) \quad \int_{\Omega} dF_2(x_2) \dots dF_n(x_n) = c_1 > 0$$

$$(2.10) \quad \min_{\Omega} P_0(x_2, \dots, x_n) = c_2 > 0$$

We can find positive constants C_3, C_4, C_5 such that the relations

$$(2.11) \quad P(x) \geq c_4 |x_1|^m$$

$$(2.12) \quad |x_1^*| \geq c_5 |x_1|$$

are satisfied, provided that $|x_1| > c_3$ and $(x_2, \dots, x_n) \in \Omega$. The integrand on the left hand side of (2.7) is nonnegative, so that

$$(2.13) \quad \begin{aligned} \int_{|x_1| > c_3} \int_{\Omega} x_1^m (|x_1| - |x_2 + \dots + x_n|)^N dF_1(x_1) \dots dF_n(x_n) &\leq \\ &\leq \sum_{p=0}^r q_p \int_{R_n} \left(\sum_{j=1}^n |x_j| \right)^{N+p} dF_1(x_1) \dots dF_n(x_n). \end{aligned}$$

Let

$$(2.14) \quad b_j = \int_{\Omega} |x_2 + \dots + x_n|^j dF_2(x_2) \dots dF_n(x_n),$$

$$(2.15) \quad I_0 = \int_{|x_1| \leq c_3} \int_{\Omega} x_1^m (|x_1| - |x_2 + \dots + x_n|)^N dF_1(x_1) \dots dF_n(x_n),$$

$$(2.16) \quad \Omega_0 = \{x \mid |x_1| \leq c_3, (x_2, \dots, x_n) \in \Omega, x \in R_n\},$$

$$(2.17) \quad c_8 = \max_{\Omega_0} \{|x_1|, ||x_1| - |x_2 + \dots + x_n||\}.$$

Now we see that

$$(2.18) \quad I_0 \leq b_0 C_8^{m+N}.$$

It follows from (2.13)—(2.18) that

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{\Omega} x_1^m (|x_1| - |x_2 + \dots + x_n|)^N dF_1(x_1) \dots dF_n(x_n) = \\ & = \sum_{j=0}^N \binom{N}{j} (-1)^j \int_{-\infty}^{\infty} |x_1|^{N+m-j} dF_1(x_1) \int_{\Omega} |x_2 + \dots + x_n|^j dF_2(x_2) \dots dF_n(x_n) = \\ (2.19) \quad & = \sum_{j=0}^N \binom{N}{j} \beta_{N+m-j}^{(1)} b_j (-1)^j \leq b_0 C_8^{m+N} + \\ & + \sum_{p=0}^r q_p \sum_{j_1^{(p)} + \dots + j_n^{(p)} = N+p} \frac{(N+p)!}{j_1^{(p)}! \dots j_n^{(p)}!} \beta_{j_1^{(p)}}^{(1)} \dots \beta_{j_n^{(p)}}^{(n)}. \end{aligned}$$

Therefore

$$\begin{aligned} & \beta_{N+m}^{(1)} b_0 \leq \sum_{j=1}^N \binom{N}{j} \beta_{N+m-j}^{(1)} b_j + b_0 C_8^{m+N} + \\ (2.20) \quad & + \sum_{p=0}^r q_p \sum_{j_1^{(p)} + \dots + j_n^{(p)} = N+p} \frac{(N+p)!}{j_1^{(p)}! \dots j_n^{(p)}!} \beta_{j_1^{(p)}}^{(1)} \dots \beta_{j_n^{(p)}}^{(n)}. \end{aligned}$$

It follows from (2.14) that there exists a positive constant b such that

$$(2.21) \quad b_j \leq b_0 b^j.$$

Thus there exists a positive constant M_0 such that

$$(2.22) \quad \beta_k^{(j)} \leq \frac{k!}{k^{1+x}} M_0^k$$

where $k=1, 2, \dots, m-1; j=1, \dots, n$ and $\alpha>0$. Assume now that (2.4) holds for $k=1, 2, \dots, N+m-1$ and for some $M \equiv M_0$ which will be selected later. Then we have to show that (2.4) is valid for $k=N+m$. Using the inequality (2.4) for $k=1, 2, \dots, N+m-1$, we obtain from (2.20) and (2.21) that

$$(2.23) \quad \beta_{N+m}^{(1)} b_0 \leq \sum_{j=1}^N \binom{N}{j} \frac{(N+m-j)!}{(N+m-j)^{1+\alpha}} M^{N+m-j} b_0 b^j + \\ + b_0 C_8^{m+N} + \sum_{p=0}^r q_p \sum_{j_1^{(p)} + \dots + j_n^{(p)} = N+p} \frac{(N+p)!}{j_1^{(p)}! \dots j_n^{(p)}!} \beta_{j_1^{(p)}}^{(1)} \dots \beta_{j_n^{(p)}}^{(n)}.$$

Note first that

$$(2.24) \quad \binom{N}{j} (m+N-j)! < \frac{(m+N)!}{j!}$$

so that

$$(2.25) \quad \beta_{N+m}^{(1)} \leq \sum_{j=1}^N \frac{(N+m)!}{j! (N+m-j)^{1+\alpha}} M^{N+m-j} b^j + \\ + C_8^{m+N} + \sum_{p=0}^r q'_p \sum_{j_1^{(p)} + \dots + j_n^{(p)} = N+p} \frac{(N+p)!}{j_1^{(p)}! \dots j_n^{(p)}!} \beta_{j_1^{(p)}}^{(1)} \dots \beta_{j_n^{(p)}}^{(n)},$$

where $q'_p = q_p |b_0|$ $|p=0, 1, \dots, r|$.

Let $0 < \alpha < \frac{3 \lg 2}{\lg 3} - 1$ then $j^{1+\alpha} \leq 2^j$ and if we apply lemma 2.1. and Schwartz's inequality, we obtain

$$(2.26) \quad \sum_{j=1}^N \frac{(N+m)!}{j! (N+m-j)^{1+\alpha}} M^{N+m-j} b^j \leq \\ \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sum_{j=1}^N \left(\frac{b}{M} \right)^j \frac{1}{j!} j^{1+\alpha} \left(\frac{N+m}{j(N+m-j)} \right)^{1+\alpha} \leq \\ \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sum_{j=1}^N \left(\frac{2b}{M} \right)^j \frac{1}{j!} \left(\frac{N+m}{j(N+m-j)} \right)^{1+\alpha} \leq \\ \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sqrt{\sum_{j=1}^N \left[\frac{1}{j!} \left(\frac{2b}{M} \right)^j \right]^2} \cdot \\ \cdot \sqrt{\sum_{j=1}^N \left(\frac{N+m}{j(N+m-j)} \right)^{2(1+\alpha)}} \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sqrt{e^{(2b/M)^2} - 1} \cdot S_2(N+m) \leq \\ \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sqrt{e^{(2b/M)^2} - 1} \cdot 2^{2+\alpha} \cdot S_2,$$

and

$$\begin{aligned}
 & \sum_{j_1^{(p)} + \dots + j_n^{(p)} = N+p} \frac{(N+p)!}{j_1^{(p)}! \dots j_n^{(p)}!} \beta_{j_1^{(p)}}^{(1)} \dots \beta_{j_n^{(p)}}^{(n)} = \\
 & = \sum_{k=1}^n \sum_{1 \leq l_1^{(p)} < \dots < l_k^{(p)} \leq n} \sum_{j_1^{(p)} + \dots + j_n^{(p)} = N+p} \frac{(N+p)! \beta_{j_1^{(p)}}^{(l_1)} \dots \beta_{j_k^{(p)}}^{(l_k)}}{j_1^{(p)}! \dots j_k^{(p)}!} \leq \\
 & \leq \frac{(N+p)! M^{N+p}}{(N+p)^{1+\alpha}} \sum_{k=1}^n \binom{n}{k} \sum_{j_1^{(p)} + \dots + j_n^{(p)} = N+p} \frac{(N+p)^{1+\alpha}}{(j_1^{(p)})^{1+\alpha} \dots (j_n^{(p)})^{1+\alpha}} = \\
 & = \frac{(N+p)! M^{N+p}}{(N+p)^{1+\alpha}} \sum_{k=1}^n \binom{n}{k} S_k(N+p) \leq \\
 (2.27) \quad & \leq \frac{(N+p)!}{(N+p)^{1+\alpha}} M^{N+p} \sum_{k=1}^n \binom{n}{k} (2^{2+\alpha} \cdot s_\alpha)^{k-1} = \\
 & = \frac{(N+p)!}{(N+p)^{1+\alpha}} M^{N+p} \frac{1}{2^{2+\alpha} \cdot s_\alpha} [(1 + 2^{2+\alpha} \cdot s_\alpha)^n - 1] = \\
 & = \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \left(\frac{N+m}{N+p} \right)^{1+\alpha} \left(\frac{1}{M} \right)^{m-p} \frac{1}{(N+p+1) \dots (N+m)} \cdot \\
 & \quad \cdot \frac{1}{2^{2+\alpha} \cdot s_\alpha} [(1 + 2^{2+\alpha} \cdot s_\alpha)^n - 1] < \\
 & < \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \left(\frac{1}{M} \right)^{m-p} \left(1 + \frac{m-p}{N+p} \right)^{1+\alpha} \frac{1}{2^{2+\alpha} \cdot s_\alpha} [(1 + 2^{2+\alpha} \cdot s_\alpha)^n - 1] \leq \\
 & \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \left(\frac{1}{M} \right)^{m-p} (1+m)^{1+\alpha}.
 \end{aligned}$$

Therefore from (2.25), (2.26) and (2.27) it follows that

$$\begin{aligned}
 (2.28) \quad \beta_{N+m}^{(1)} & \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \left[\sqrt{e^{(2b/M)^2} - 1} \cdot 2^{2+\alpha} \cdot s_\alpha + \right. \\
 & \quad \left. + \left(\frac{C_8}{M} \right)^{m+N} + \sum_{p=0}^r q'_p \left(\frac{1}{M} \right)^{m-p} (1+m)^{1+\alpha} \right].
 \end{aligned}$$

Since the quantities $b, \alpha, s_\alpha, c_8, m, q'_p, p$ are independent of N , we can find a sufficiently large $M \geq c_8$ such that

$$(2.29) \quad \sqrt{e^{(2b/M)^2} - 1} \cdot 2^{2+\alpha} \cdot s_\alpha + \left(\frac{C_8}{M} \right)^m + \sum_{p=0}^r q'_p \left(\frac{1}{M} \right)^{m-p} (1+m)^{1+\alpha} < 1.$$

Then

$$\beta_{N+m}^{(1)} \leq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m}.$$

If N (and thus k) is an odd number then we use instead of (2.7) the inequality

$$(2.30) \quad \int_{R_n} P(x) (|x_1| - |x_2 + \dots + x_n|)^{N-1} |x_1 + \dots + x_n| dF_1(x_1) \dots dF_n(x_n) \cong \\ \cong \sum_{p=0}^r |\gamma_p| \int_{R_n} \left(\sum_{j=1}^n |x_j| \right)^{N+p} dF_1(x_1) \dots dF_n(x_n)$$

and we see easily that inequality (2.4) is valid for odd k . This completes the proof of the inequality (2.4).

Hence from (2.4)

$$(2.31) \quad \lim_{k \rightarrow \infty} \left(\frac{\beta_k^{(1)}}{k!} \right)^{1/k} \cong M$$

therefore the characteristic function $f_1(t)$ is regular in the circle of radius $\frac{1}{M}$. It follows that the characteristic function $f_1(z)$ $|z=t+iv, t \in R_1, v \in R_1|$ is regular at least in the strip $|\operatorname{Im} z| < \frac{1}{M}$.

Next we show that the regularity of $f_1(z)$ in a strip $|\operatorname{Im} z| < V$ $|V > 0|$ implies that $f_1(z)$ is also regular in the strip $|\operatorname{Im} z| < V + \frac{1}{M}$. The statement of theorem 2.1. then follows by induction.

Assume that $f_1(z)$ is regular in the strip $|\operatorname{Im} z| < V$ and observe that equation (2.1) holds for all complex z in this strip, so that $|x=(x_j) \in R_n|$

$$(2.32) \quad \int_{R_n} P(x) e^{iz(xa^*)} dF_1(x_1) \dots dF_n(x_n) = \\ = \sum_{p=0}^r \gamma_p \int_{R_n} (xa^*)^p e^{iz(xa^*)} dF_1(x_1) \dots dF_n(x_n).$$

By differentiating (2.32) N -times with respect to z and then putting $z = -iv_0$ we obtain

$$(2.33) \quad \int_{R_n} P(x) (xa^*)^N e^{v_0(xa^*)} dF_1(x_1) \dots dF_n(x_n) = \\ = \sum_{p=0}^r \gamma_p \int_{R_n} (xa^*)^{N+p} e^{v_0(xa^*)} dF_1(x_1) \dots dF_n(x_n)$$

for $|v_0| < V$. Introduce now the functions

$$(2.34) \quad G_j(x) = A_j^{-1} \int_{-\infty}^x e^{v_j y} dF_j(y) \quad |j = 1, 2, \dots, n|$$

where

$$(2.35) \quad A_j = \int_{-\infty}^{\infty} e^{v_0 y} dF_j(y)$$

Clearly $G_j(x)$ is a distribution function. We denote its characteristic function by $g_j(t)$. We divide both sides of the equation (2.33) by the product $A_1 \dots A_n$ and we see that

$$(2.36) \quad \begin{aligned} & \int_{R_n} P(x) (xa^*)^N dG_1(x_1) \dots dG_n(x_n) = \\ & = \sum_{p=0}^r \gamma_p \int_{R_n} (xa^*)^{N+p} dG_1(x_1) \dots dG_n(x_n). \end{aligned}$$

This relation is identical to (2.2). We proceed in the same way as in the first part of the proof of theorem 2.1., to show that $g_1(z)$ is also an analytic characteristic function which is regular in the strip $|\operatorname{Im} z| < \frac{1}{M}$. Since

$$g_1(z) = A_1^{-1} \int_{-\infty}^{\infty} e^{izx + v_0 x} dF_1(x),$$

it follows immediately that the characteristic function $f_1(z)$ is regular in the strip $|\operatorname{Im} z| < V + \frac{1}{M}$, and so our theorem 2.1. is completely proved.

As extension of theorem 2.1. it is not difficult to prove:

Theorem 2.2. *Let the components of the random vector-variable $\zeta = (\zeta_j) \in R_n$ be independent but not necessarily identically distributed random variables. Let the distribution function of ζ_j be denoted by $F_j(x)$ and assume that ζ_j has moments of all orders $|j=1, 2, \dots, n|$. Let*

$$A = \zeta \alpha^*, \quad \alpha = (\alpha_j) \in R_n, \quad \alpha_j \neq 0 \quad |j=1, \dots, n|$$

and let

$$P = \sum A_{v_1 \dots v_n} \zeta_1^{v_1} \dots \zeta_n^{v_n}$$

be a common polynomial statistic of degree p_1 and order m $|m \leq p_1|$. If P has polynomial regression of order r $|0 \leq r \leq m-1|$ on A then the characteristic function $f_j(t)$ of $F_j(x)$ is an entire characteristic function $|j=1, \dots, n|$.

PROOF. By the assumption we see, similarly to (2.1), that

$$(2.37) \quad \begin{aligned} & \int_{R_n} P(x) e^{it \sum_{j=1}^n \alpha_j x_j} dF_1(x_1) \dots dF_n(x_n) = \\ & = \sum_{p=0}^r \gamma_p \int_{R_n} (x \alpha^*)^p e^{it(x \alpha^*)} dF_1(x_1) \dots dF_n(x_n). \end{aligned}$$

The transformation $y_j = \alpha_j x_j$ $|j=1, \dots, n|$ yields

$$(2.38) \quad \begin{aligned} & \int_{R_n} P_1(y) e^{it(y a^*)} dG_1(y_1) \dots dG_n(y_n) = \\ & = \sum_{p=0}^r \int_{R_n} (y a^*)^p e^{it(y a^*)} dG_1(y_1) \dots dG_n(y_n). \end{aligned}$$

Here $G_j(x)$ is the distribution function of $\eta_j = \alpha_j \xi_j \mid j=1, \dots, n \mid$ and

$$P_1(y) = P\left(\frac{y_1}{\alpha_1}, \dots, \frac{y_n}{\alpha_n}\right)$$

is also a common polynomial statistic. We conclude in the same way as in the proof of the theorem 2.1., that the characteristic function of $G_j(x)$ and also the characteristic function of $F_j(x)$ are entire functions ($j=1, \dots, n$).

Let

$$P = \sum A_{v_1 \dots v_n} \xi_1^{v_1} \dots \xi_n^{v_n}$$

be a polynomial statistic of degree p_1 . Suppose that no exponent in P exceeds m while at least one variable has exponent $m \mid m \leq p_1 \mid$. We form the polynomial statistic

$$P^* = \frac{1}{n!} \sum^* \sum A_{v_1 \dots v_n} \xi_{k_1}^{v_1} \dots \xi_{k_n}^{v_n}$$

where the first summation runs over all permutations $k_1, \dots, k_n$ of the first n integers, while the second summation is taken over all subscripts satisfying (1.1). The statistic P^* is said to be the adjoint polynomial statistic of P .

Theorem 2.3. *Let the components of the random vector variable $\xi = (\xi_j) \in R_n$ be independent and identically distributed with $F(x)$. Assume that $F(x)$ has moments of all orders. Let $\Lambda = \xi \alpha^*$, $\alpha = (\alpha_j) \in R_n$, $\alpha_j \neq 0$ and let*

$$P = \sum A_{v_1 \dots v_n} \xi_1^{v_1} \dots \xi_n^{v_n}$$

be a regular polynomial statistic of degree p_1 and order $m \mid m \leq p_1 \mid$. If P has polynomial regression of order $r \mid 0 \leq r \leq m-1 \mid$ on Λ then P^ (the adjoint polynomial statistic of P) has polynomial regression of order $r \mid 0 \leq r \leq m-1 \mid$ on Λ .*

PROOF. Since P has polynomial regression of order $r \mid 0 \leq r \leq m-1 \mid$ on Λ , therefore

$$\begin{aligned} (2.39) \quad & \int_{R_n} P(x) e^{it(x\alpha^*)} dF_1(x_1) \dots dF_n(x_n) = \\ & = \sum_{p=0}^r \gamma_p \int_{R_n} (x\alpha^*)^p e^{it(x\alpha^*)} dF_1(x_1) \dots dF_n(x_n). \end{aligned}$$

By the transformation

$$x_j = \frac{1}{\alpha_j} y_j \quad |j=1, \dots, n|$$

we get the expression (2.38) and then by the transformation $y_j = y_{k_j} \mid j=1, \dots, n \mid$ the formula

$$\begin{aligned} (2.40) \quad & \int_{R_n} P(y_{k_1}, \dots, y_{k_n}) e^{it(y\alpha^*)} dG_1(y_{k_1}) \dots dG_n(y_{k_n}) = \\ & = \sum_{p=0}^r \gamma_p \int_{R_n} (y\alpha^*)^p e^{it(y\alpha^*)} dG_1(y_{k_1}) \dots dG_n(y_{k_n}), \end{aligned}$$

follows, where $k_1, \dots, k_n$ is a permutation of the first n integers. Now the transformation $y_j = \alpha_j x_j \mid j=1, \dots, n$ yields

$$(2.41) \quad \begin{aligned} & \int_{R_n} P(x_{k_1}, \dots, x_{k_n}) e^{it(xz^*)} dF(x_1) \dots dF(x_n) = \\ & = \sum_{p=0}^r \gamma_p \int_{R_n} (xz^*)^p e^{it(xz^*)} dF(x_1) \dots dF(x_n) \end{aligned}$$

i.e. $P(\xi_{k_1}, \dots, \xi_{k_n})$ and thus P^* has also polynomial regression of order $r \mid 0 \leq r \leq m-1$ on Λ .

Theorem 2.4. *Let the components of $\xi = (\xi_j) \in R_n$ be independently and identically distributed random variables with distribution function $F(x)$. Assume that $F(x)$ has moments of all orders. Let $\Lambda = \xi \alpha^*$ where $\alpha = (\alpha_j) \in R_n, \alpha_j \neq 0$ and let*

$$P = \sum A_{v_1 \dots v_n} \xi_1^{v_1} \dots \xi_n^{v_n}$$

be a polynomial statistic of degree p_1 and order $m \mid m \leq p_1$. Suppose that no exponent in P exceeds m while at least one variable has exponent m . Let the adjoint polynomial of P be a nonnegative polynomial. If P has polynomial regression of order $r \mid 0 \leq r \leq m-1$ on Λ then the characteristic function of $F(x)$ is an entire function.

PROOF. The adjoint polynomial P^* of P is nonnegative therefore P^* is a regular polynomial statistic. On the basis of theorem 2.3. we obtain that our theorem 2.4. is a consequence of theorem 2.2.

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