

Absolute total-effective Nörlund method

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1. Introduction and the main result

Let $\{p_n\}$ be a sequence of real numbers such that P_n^0 or $P_n = \sum_{k=0}^n p_k \neq 0$ and $P_n^1 = \sum_{k=0}^n P_k \neq 0$. The Nörlund method (N, p_n) is associated with the series to sequence transformation which transforms a series $\sum a_n$ to the sequence $\{t_n^p(a)\}$ defined by

$$t_n^p(a) = \frac{1}{P_n} \sum_{k=0}^n P_{n-k} a_k.$$

A sequence $\{s_n\} \in BV$, if $\sum_{n=1}^{\infty} |s_n - s_{n-1}| < \infty$; and $\{s_n\} \in B$, if $\{s_n\}$ is a bounded sequence.

The object of the present paper is to improve the theorem proved in [3] and we shall throughout use the definitions and notations contained therein. We shall prove the following.

Theorem. *If the sequence $\{p_n\}$ is such that*

$$(1.1) \quad \{R_n\} \equiv \{(n+1)p_n/P_n\} \in BV \quad \text{and} \quad S_n^* = \left\{ \frac{1}{P_n} \sum_{k=0}^n \frac{|P_k|}{k+1} \right\} \in B,$$

then the (N, P_n) method is absolute total-effective.

Writing $P_n^{1*} = \sum_{k=0}^n |P_k|$, P_n^* or $P_n^{0*} = \sum_{k=0}^n |p_k|$ and $\{R_n^1\} \equiv \{(n+1)P_n/P_n^1\}$, the above result under the additional conditions viz.,

$$(1.2) \quad P_n^{j*} = O(|P_n^j|), \quad j = 0, 1 \quad \text{and} \quad \{R_n^1\} \in BV,$$

was proved in [3]. From the theorem proved here we shall deduce in the last section the following which includes *inter alia* some earlier results and provides a shorter alternative proof of them (see e.g. [2]).

Corollary. *The $(C, 1)(N, p_n)$ method is absolute total-effective, if $P_n > 0$ and (1.1) holds.*

2. Some preliminary results

Lemma 1. Let $\{a_k\}$ be a given sequence, then for any x , we have

$$(1-x) \sum_{k=m}^n a_k x^k = - \sum_{k=m}^n \Delta a_{k-1} x^k + a_{m-1} x^m - a_n x^{n+1},$$

where m, n are integers such that $0 \leq m \leq n$ and $\Delta a_k = a_k - a_{k+1}$.

The proof of Lemma 1 is direct.

Lemma 2. If $\{R_n\} \in B$ and $\{S_n^*\} \in B$, then $P_n^* \leq K|P_n|$, $\{1/R_n^1\} \in B$,

$$(2.1) \quad |P_k| \sum_{n=k}^{\infty} \frac{1}{(n+1)|P_n|} \leq K (k = 0, 1, 2, \dots),$$

and $|P_n| \rightarrow \infty$, as $n \rightarrow \infty$.

PROOF. That $P_n^* \leq K|P_n|$ follows directly from the hypotheses, when we observe that

$$P_n^* = \sum_{k=0}^n |R_k| \frac{|P_k|}{k+1} \leq K \sum_{k=0}^n \frac{|P_k|}{k+1} \leq K|P_n|.$$

Using this result and observing that

$$|P_n^1| \leq \sum_{k=0}^n P_k^* \leq (n+1)P_n^* \leq K(n+1)|P_n|,$$

we prove that $\{1/R_n^1\} \in B$. For the proof of (2.1) reference may be made to ([4], pp. 13—14). Finally, since $P_n^* \leq K|P_n|$, the convergence of the series in (2.1) leads to the result that $|P_n| \rightarrow \infty$, as $n \rightarrow \infty$.

Lemma 3. If $p_n = o(P_n)$ and $|P_n| \rightarrow \infty$, as $n \rightarrow \infty$, then $P_n^{1*} \leq K|P_n^1|$ for all n .

PROOF. Since $p_n = o(P_n)$, for a given positive $\varepsilon < 1$, we have a positive integer N , such that $-\varepsilon < p_n/P_n < \varepsilon$, whenever $n > N$. Writing $P_{n-1}/P_n = 1 - p_n/P_n$, we observe that for $n > N$, $0 < 1 - \varepsilon < P_{n-1}/P_n < 1 + \varepsilon$ (cf. proof of Lemma 2 in [6]) and this implies that $P_N, P_{N+1}, \dots$, are all positive or all negative. Since N is fixed and $P_n^1 \neq 0$, we have $P_k^{1*} \leq K|P_k^1|$, for $0 \leq k \leq N$. If $k > N$, $P_k^{1*} = P_N^{1*} + \delta(P_k^1 - P_N^1)$, where $\delta = -1$ or $+1$, according as $P_{N+1}, P_{N+2}, \dots$, are all negative or all positive. Thus, $P_k^{1*} < K + K|P_k^1|$ and the result of the lemma follows, when we observe that $P_n^1 \neq 0$ for all n , and the sequence $\{P_n\}$ is ultimately a positive sequence or a negative sequence and by hypothesis $|P_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Lemma 4. If $P_n^{1*} \leq K|P_n^1|$ for all n , and $\{R_n\} \in B$, then $\{R_n^1\} \in B$. Further (2.1) implies the following, whenever $\{R_n^1\} \in B$,

$$(2.2) \quad |P_k| \sum_{n=k}^{\infty} \frac{1}{|P_n^1|} \leq K (k = 0, 1, 2, \dots).$$

PROOF. We write

$$(n+1)P_n = - \sum_{k=0}^n \Delta \{kP_{k-1}\} + P_0 = \sum_{k=0}^n P_{k-1} + \sum_{k=0}^n R_k P_k + P_0$$

and, therefore, since $\{R_n\} \in B$,

$$(n+1)|P_n| \leq K P_n^{1*} \leq K |P_n^1|,$$

by virtue of the hypothesis that $P_n^{1*} \leq K |P_n^1|$. Thus $\{R_n^1\} \in B$. The remaining part of the lemma follows, when we write

$$\sum_{n=k}^{\infty} \frac{1}{|P_n^1|} = \sum_{n=k}^{\infty} \frac{|R_n^1|}{(n+1)|P_n^1|}.$$

Lemma 5. *If $P_n^* \leq K |P_n|$ for all n , then uniformly in $0 < t \leq \pi$*

$$\left| \sum_{k=0}^n P_k \exp ikt \right| \leq K t^{-1} |P_n|.$$

Lemma 4 is essentially the same as Lemma 3 in [4].

Lemma 6. *Let $q_n > 0$, $Q_n = \sum_{k=0}^n q_k$ and $d_n = \sum_{k=0}^n q_k c_k / Q_n$, then if the sequence $\{c_n\} \in BV$, the sequence $\{d_n\} \in BV$.*

Lemma 6 is due to MOHANTY ([5], Lemma 4).

Lemma 7. *If $\{R_n\} \in B$ and $P_n^{1*} \leq K |P_n^1|$, then $|N, p_n| \subset |N, P_n|$, i.e. every series summable $|N, p_n|$ is also summable $|N, P_n|$.*

PROOF. We first observe that $\{1/P_n^1\} \in BV$. For,

$$\sum_{n=1}^{\infty} \left| \frac{1}{P_{n-1}^1} - \frac{1}{P_n^1} \right| \leq K \sum_{n=1}^{\infty} \frac{|P_n|}{P_n^{1*} P_{n-1}^1} = K \sum_{n=1}^{\infty} \left(\frac{1}{P_{n-1}^{1*}} - \frac{1}{P_n^{1*}} \right) \leq K,$$

since $P_n^{1*} \leq K |P_n^1|$ and $\{P_n^{1*}\}$ is positive and monotonic increasing.

Since the hypothesis $\{R_n\} \in B$, implies that $p_n = o(P_n)$, $n \rightarrow \infty$, it follows from the proof of Lemma 3 that there exists a positive integer N such that $P_N, P_{N+1}, \dots$, are all positive or all negative. We now write n -th (N, P_n) mean of $\sum a_n$ as (cf. [1], p. 360)

$$\begin{aligned} t_{n+N}^P(a) &= \frac{1}{P_{n+N}^1} \sum_{k=0}^{N-1} P_k t_k^P(a) + \frac{\sum_{k=0}^n P_{N+k} t_{N+k}^P(a)}{P_{N+n}^1 - P_{N-1}^1} \left(1 - \frac{P_{N-1}^1}{P_{N+n}^1} \right) = \\ &= \frac{K}{P_{n+N}^1} + \mu_n(a) \left(1 - \frac{P_{N-1}^1}{P_{n+N}^1} \right) \end{aligned}$$

say. Since $\{1/P_n^1\} \in BV$, in order to prove that $\{t_n^P(a)\} \in BV$, it is sufficient to show that $\{\mu_n(a)\} \in BV$. However, if $\sum a_n$ is summable $|N, p_n|$, then $\{t_n^P(a)\} \in BV$ and by Lemma 6, $\{\mu_n(a)\} \in BV$, since $P_N, P_{N+1}, \dots$, are all positive or all negative.

Lemma 8. If $\theta(t) \in BV(0, \pi)$ and (1.1) holds then the series

$$\sum_n \int_0^\pi \Theta(t) \cos nt \, dt$$

is summable $[N, p_n]$.

Lemma 8 follows from the proof of Theorem C in [4], when we appeal to the Lemma 2 of the present paper.

3. Proof of the Theorem

Integrating by parts, we have for the Fourier series $L(x)$ (cf. [3])

$$\begin{aligned} \frac{\pi}{2} a_k &= \int_0^\pi \varphi(t) \cos kt \, dt = \int_0^\pi \left\{ t \cos kt - \int_0^t \cos ku \, du \right\} d\varphi_1(t) = \\ &= \int_0^\pi t \cos kt \, d\varphi_1(t) + \int_0^\pi \varphi_1(t) \cos kt \, dt = u_k + v_k, \end{aligned}$$

say. Since x is $|F_1|$ -regular, $\varphi_1(t) \in BV(0, \pi)$ and $\sum_n v_n$ is summable $[N, p_n]$ by Lemma 8. The $[N, P_n]$ summability of $\sum_n v_n$ now follows, when we appeal to Lemma 7. Considering the series $\sum_n u_n$, we observe that (cf. [3])

$$t_n^P(u) - t_{n-1}^P(u) = \int_0^\pi \left\{ \frac{t}{P_n^1 P_{n-1}^1} \sum_{k=0}^n (P_n^1 P_k - P_k^1 P_n) \cos(n-k)t \right\} d\varphi_1(t).$$

Since $\int_0^\pi |d\varphi_1(t)| \leq K$, in order to show that $\{t_n^P(u)\} \in BV$, it is sufficient to demonstrate that uniformly in $0 < t \leq \pi$

$$(3.1) \quad \Sigma = t \sum_{n=1}^\infty \left| \frac{1}{P_n^1 P_{n-1}^1} \sum_{k=0}^n (P_n^1 P_k - P_k^1 P_n) \exp ikt \right| \leq K.$$

By virtue of Lemma 1, we have

$$\sum_{k=0}^n (P_n^1 P_k - P_n P_k^1) \exp ikt = (1 - \exp it)^{-1} \sum_{k=0}^n (P_n^1 P_k - P_n P_k) \exp ikt.$$

Thus,

$$\begin{aligned} (3.2) \quad \Sigma &\leq t \sum_{n=1}^\tau \left| \frac{1}{P_n^1 P_{n-1}^1} \right| \sum_{k=0}^n |P_n^1 P_k - P_n P_k^1| + \\ &+ K \sum_{n=\tau+1}^\infty \left| \frac{1}{P_n P_{n-1}} \right| \sum_{k=0}^{\tau-1} |P_n^1 P_k - P_n P_k| + \\ &+ K \sum_{n=\tau+1}^\infty \left| \frac{1}{P_{n-1}^1} \sum_{k=\tau}^n \left(\frac{P_k}{P_k} - \frac{P_n}{P_n^1} \right) P_k \exp ikt \right| = \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned}$$

say. Since by Lemma 3, $P_n^{1*} \leq K |P_n^1|$, we have

$$(3.3) \quad \begin{aligned} \Sigma_1 &\leq t \sum_{n=1}^{\tau} \frac{1}{|P_{n-1}^1|} \sum_{k=0}^n |P_k| + t \sum_{n=1}^{\tau} \left| \frac{P_n}{P_n^1 P_{n-1}^1} \right| \sum_{k=0}^{n-1} P_k^{1*} \leq \\ &\leq Kt \sum_{n=1}^{\tau} \left| \frac{P_n^1}{P_{n-1}^1} \right| + Kt \sum_{n=1}^{\tau} |R_n^1| \leq K. \end{aligned}$$

by virtue of Lemmas 2—4.

Again using the result that $P_n^{1*} \leq K |P_n^1|$, we have from Lemma 4,

$$(3.4) \quad \begin{aligned} \Sigma_2 &\leq K |P_{\tau-1}| \sum_{n=\tau+1}^{\infty} \frac{1}{|P_{n-1}^1|} + K \sum_{n=\tau+1}^{\infty} \left| \frac{P_n}{P_n^1 P_{n-1}^1} \right| \sum_{k=0}^{\tau-1} P_k^* \leq \\ &\leq K + K\tau P_{\tau-1}^* \sum_{n=\tau+1}^{\infty} \frac{|R_n^1|}{n |P_{n-1}^1|} \leq K, \end{aligned}$$

since $P_n^* \leq K |P_n|$, by virtue of Lemma 2.

Applying Abel's transformation, we write

$$(3.5) \quad \begin{aligned} \Sigma_3 &\leq K \sum_{n=\tau+1}^{\infty} \left| \frac{1}{P_{n-1}^1} \sum_{k=\tau}^{n-1} \Delta \left(\frac{R_k}{k+1} \right) \sum_{v=\tau}^k P_v \exp ivt \right| + \\ &+ K \sum_{n=\tau+1}^{\infty} \left| \frac{1}{P_{n-1}^1} \left(\frac{P_n}{P_n} - \frac{P_n}{P_n^1} \right) \sum_{v=\tau}^n P_v \exp ivt \right| \leq \\ &\leq K\tau \sum_{n=\tau+1}^{\infty} \frac{1}{|P_{n-1}^1|} \sum_{k=\tau}^{n-1} \left\{ \frac{|\Delta R_k|}{k+1} + \frac{|R_{k+1}|}{k(k+1)} \right\} |P_k| + \\ &+ K\tau \sum_{n=\tau+1}^{\infty} \frac{|R_n R_n^1|}{(n+1)^2} + K\tau \sum_{n=\tau+1}^{\infty} \frac{|R_n^1 R_{n-1}^1|}{(n+1)^2} \leq \\ &\leq K\tau \sum_{k=\tau}^{\infty} \frac{|\Delta R_k|}{k} |P_k| \sum_{n=k+1}^{\infty} \frac{1}{|P_{n-1}^1|} + \\ &+ K\tau \sum_{k=\tau}^{\infty} \frac{1}{k(k+1)} |P_k| \sum_{n=k+1}^{\infty} \frac{1}{|P_{n-1}^1|} + K \leq \\ &\leq K \sum_{k=\tau}^{\infty} |\Delta R_k| + K \leq K, \end{aligned}$$

by virtue of Lemmas 2—5 and the hypothesis that $\{R_n\} \in BV$, which implies that $P_n/P_{n-1} \rightarrow 1$, as $n \rightarrow \infty$.

Combining (3.2)–(3.5), we prove (3.1) and thus $\sum_n u_n$ is summable $[N, P_n]$.

This completes the proof of the $|F_1|$ -effectiveness part of the theorem.

The rest of the theorem follows from the foregoing proof of the $|F_1|$ -effectiveness, when we refer to section 5 of [3].

4. Proof of the Corollary

We need the following additional lemma.

Lemma 9. *If $P_n > 0$, $P_n \rightarrow \infty$, $\{R_n^1\} \in B$ and $\{1/R_n^1\} \in B$, then the $|N, P_n|$ method is equivalent to the $|(C, 1)(N, p_n)|$ method.*

Lemma 9 follows directly from Theorems 1 and 2 of [1]. Now we see that if $P_n > 0$ and (1.1) holds then by virtue of Lemmas 2 and 4 of the present paper, the hypotheses of Lemma 9 are satisfied and in conclusion we obtain the result of the Corollary from the theorem already proved.

References

- [1] G. DAS, Tauberian theorems for absolute Nörlund summability, *Proc. London Math. Soc.* (3) **19** (1969), 357—384.
- [2] H. P. DIKSHIT, Absolute $(C, 1)(N, p_n)$ summability of a Fourier series and its conjugate series, *Pacific J. Math.* **26** (1968), 245—256.
- [3] H. P. DIKSHIT, Absolute total-effective triangular matrix method, *Math. Ann.* **186** (1970), 101—114.
- [4] H. P. DIKSHIT, Absolute summability of some series related to a Fourier series, *J. Australian Math. Soc.* **13** (1971), 7—14.
- [5] R. MOHANTY, A criterion for the absolute convergence of a Fourier series, *Proc. London Math. Soc.* **41** (1949), 186—196.
- [6] A. PEYERIMHOFF, On convergence fields of Nörlund means, *Proc. Amer. Math. Soc.* **7** (1956), 335—347.

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