

Integrability conditions of a structure f_λ satisfying $f^3 - \lambda^2 f = 0$

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Summary: **Yano** [2] has obtained certain results on a structure defined by a tensor field f ($f \neq 0$) of type (1,1) satisfying $f^3 + f = 0$. **Ishihara** and **Yano** [1] have obtained its integrability conditions. Now we propose to obtain integrability conditions of a structure f_λ satisfying $f^3 - \lambda^2 f = 0$, where λ is a complex number not equal to zero.

In section 2, we have studied the Nijenhuis tensor of f_λ -structure and deduced some of its properties. In sections 3 and 4, we have obtained the conditions of partial integrability and integrability of this structure in terms of its Nijenhuis tensor.

1. Preliminaries: Let M^n be an n -dimensional differentiable manifold of class C^∞ and let there be given a (1,1) tensor field f ($f \neq 0$) of class C^∞ satisfying:

$$(1.1) \quad f^3 - \lambda^2 f = 0,$$

where λ is a complex number not equal to zero.

Let us define the operators s and t by

$$(1.2) \quad s = \left(\frac{f}{\lambda}\right)^2, \quad t = I - \left(\frac{f}{\lambda}\right)^2,$$

I denoting the identity operator. Then we have

Theorem (1.1). *For a tensor field $f \neq 0$ satisfying (1.1), the operators s and t defined by (1.2) and applied to the tangent space at a point of the manifold are complementary projection operators.*

PROOF. By virtue of (1.1) and (1.2), we have

$$\begin{aligned} s + t &= I. \\ s^2 &= \left(\frac{f}{\lambda}\right)^4 = \frac{\lambda^2 f^2}{\lambda^4} = \left(\frac{f}{\lambda}\right)^2 = s. \\ t^2 &= I + \left(\frac{f}{\lambda}\right)^4 - 2\left(\frac{f}{\lambda}\right)^2 = 1 - \left(\frac{f}{\lambda}\right)^2 = t. \\ st &= ts = \left(\frac{f}{\lambda}\right)^2 - \left(\frac{f}{\lambda}\right)^4 = 0. \end{aligned}$$

Which proves the theorem.

Let S and T be the complementary distributions corresponding to the projection operators s and t respectively. If f is of constant rank r , then the dimensions of S and T are r and $(n-r)$ respectively. We call such a structure a f_λ -structure of rank r and the manifold M^n with this structure a f_λ -manifold.

Theorem (1.2). For a tensor field $f \neq 0$ satisfying (1.1) and the operators s and t defined by (1.2), we have

$$(1.3) \quad \begin{aligned} fs &= sf = f, & ft &= tf = 0. \\ f^2s &= \lambda^2s, & f^2t &= tf^2 = 0. \end{aligned}$$

i.e. f acts on S as a π -structure operator and on T as a null operator.

PROOF. By virtue of (1.1) and (1.2), we have

$$\begin{aligned} fs &= sf = f \left(\frac{f}{\lambda} \right)^2 = \frac{f^3}{\lambda^2} = f. \\ f^2s &= f(fs) = f^2 = \lambda^2s. \\ ft &= tf = f - f \left(\frac{f}{\lambda} \right)^2 = 0. \\ f^2t &= tf^2 = f(ft) = 0. \end{aligned}$$

Hence the result.

Corollary (1.1): The f_λ -structure of maximal rank is a π -structure.

PROOF. If the rank of $f=n$, then $\dim T=0$ and $\dim S=n$. In this case, $t=0$ and $s=I$.

Hence f satisfies:

$$I - (f/\lambda)^2 = 0,$$

i.e.

$$f^2 - \lambda^2I = 0.$$

Hence the result.

2. Nijenhuis tensor of f_λ -structure:

For convenience henceforth, we will write f in place of f_λ -structure. Then the Nijenhuis tensor $N(X, Y)$ of f is

$$(2.1) \quad N(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y],$$

i.e.

$$(2.2) \quad N(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + \lambda^2s[X, Y],$$

in consequence of (1.2).

Equation (2.1) can also be written in the form

$$(2.3) \quad N(X, Y) = N(sX, sY) + N(sX, tY) + N(tX, sY) + N(tX, tY),$$

in consequence of (1.3).

Theorem (2.1). We have the following identities:

$$(2.4) \quad N(tX, tY) = s \cdot N(tX, tY) = \lambda^2 s[tX, tY],$$

$$(2.5) \quad t \cdot N(X, Y) = t \cdot [fX, fY],$$

$$(2.6) \quad t \cdot N(sX, sY) = t \cdot [fX, fY],$$

$$(2.7) \quad t \cdot N(fX, fY) = \lambda^4 t \cdot [sX, sY].$$

PROOF. The proofs of (2.4) to (2.7) follow by virtue of the equations (1.2), (1.3) and (2.2).

Theorem (2.2). For any vector fields X and Y , the following conditions are equivalent:

$$(i) \quad t \cdot N(X, Y) = 0,$$

$$(ii) \quad t \cdot N(sX, sY) = 0,$$

$$(iii) \quad t \cdot N(fX, fY) = 0.$$

PROOF. In consequence of (1.2), (1.3) and (2.2), it can be easily proved that

$$N(sX, sY) = 0 \quad \text{if and only if} \quad N(fX, fY) = 0,$$

for all vector fields X and Y .

Thus, by virtue of (2.5) and (2.6), the conditions (i), (ii) and (iii) are equivalent to each other.

The Lie derivative $L_Y f$ of the tensor field f with respect to a vector field Y is, by definition, a tensor field of the same type as f given by

$$(2.8) \quad (L_Y f)X = f[X, Y] - [fX, Y].$$

Then by virtue of (1.3), (2.1) and (2.8), we have

$$(2.9) \quad N(sX, tY) = f(L_{tY} f)sX = f\{s(L_{tY} f)sX\},$$

i.e.

$$(2.10) \quad f \cdot N(sX, tY) = \lambda^2 s(L_{tY} f)sX,$$

in consequence of (1.2).

3. Integrability conditions:

Theorem (3.1). For any two vector fields X and Y , the distribution T is integrable if and only if

$$(3.1) \quad N(tX, tY) = 0,$$

which is equivalent to

$$(3.2) \quad s \cdot N(tX, tY) = 0.$$

PROOF. The distribution T is integrable if and only if

$$s \cdot [tX, tY] = 0,$$

for any two vector fields X and Y . Thus, by virtue of (2.4), the theorem follows.

Theorem (3.2). *The distribution S is integrable if and only if any one of the conditions of theorem (2.2) is satisfied.*

PROOF. The distribution S is integrable if and only if

$$t \cdot [sX, sY] = 0,$$

for any two vector fields X and Y . Therefore, by virtue of (2.7), the theorem follows.

Theorem (3.3). For any two vector fields X and Y , the distributions S and T are both integrable if and only if

$$(3.3) \quad N(X, Y) = s \cdot N(sX, sY) + N(sX, tY) + N(tX, sY).$$

PROOF. By virtue of $s+t=I$, equation (2.3) can also be written as

$$(3.4) \quad N(X, Y) = s \cdot N(sX, sY) + t \cdot N(sX, sY) + N(sX, tY) + \\ + N(tX, sY) + N(tX, tY).$$

Thus by virtue of (3.1), (3.4) and theorem (3.2), we get the result.

Now suppose that the distribution S is integrable and take an arbitrary vector field X_1 tangent to an integral manifold of S . Let us define the operator f_* as follows:

$$f_* X_1 = fX_1.$$

Thus by virtue of theorem (1.2), the induced structure f_* is a π -structure on each integral manifold of S .

Let us denote by $N_*(X_1, Y_1)$ the vector valued 2-form corresponding to the Nijenhuis tensor of the π -structure induced from f_λ -structure on each integral manifold of S and for any two vector fields X_1 and Y_1 tangent to an integral manifold of S . Then we have

$$(3.5) \quad N_*(X_1, Y_1) = [f_* X_1, f_* Y_1] - f_* [f_* X_1, Y_1] - \\ - f_* [X_1, f_* Y_1] + \lambda^2 [X_1, Y_1];$$

from which in view of (2.1), it follows that

$$(3.6) \quad N(sX, sY) = N_*(sX, sY)$$

for any two vector fields X and Y in the manifold.

Definition (3.1): We say that the f_λ -structure is *partially integrable* if the distribution S is integrable and the π -structure f_* induced from f on each integral manifold of S is also integrable.

Theorem (3.4). For any two vector fields X and Y , a necessary and sufficient condition for the f_λ -structure to be partially integrable is that

$$(3.7) \quad N(sX, sY) = 0,$$

which is equivalent to

$$(3.8) \quad N(fX, fY) = 0.$$

PROOF. For any two vector fields X and Y , $N(sX, sY) = 0$ if and only if $N(fX, fY) = 0$. Therefore, by virtue of (3.6) and theorem (3.2), we obtain the result.

Theorem (3.5). For any two vector fields X and Y , a necessary and sufficient condition for the distribution T to be integrable and the f_λ -structure to be partially integrable is that

$$(3.9) \quad N(X, Y) = N(sX, tY) + N(tX, sY).$$

PROOF. The proof of the theorem follows by virtue of the equations (2.3), (3.1) and (3.7).

4. The condition $N(sX, tY) = 0$.

Theorem (4.1). The tensor field $s(L_{tY}f)s$ vanishes identically for any vector field Y if and only if

$$(4.1) \quad N(sX, tY) = 0,$$

for any two vector fields X and Y .

PROOF. The proof of the theorem follows by virtue of the equations (2.9) and (2.10).

When both distributions S and T are integrable, we can choose a local coordinate system such that all S are represented by putting $(n-r)$ local coordinates constant and all T by putting the other r coordinates constant. Such a coordinate system will be called an 'adapted coordinate system'.

It can be supposed that in an adapted coordinate system, the projection operators s and t have the components of the form:

$$(4.2) \quad s = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

respectively, where I_r is a unit matrix of order r and I_{n-r} is of order $(n-r)$.

Since f satisfies: $fs = sf = f$ and $ft = tf = 0$, the tensor f has the components of the form

$$(4.3) \quad f = \begin{pmatrix} f_r & 0 \\ 0 & 0 \end{pmatrix}$$

in an adapted coordinate system, where f_r is a square matrix of order $r \times r$.

Thus, the Lie derivative $L_{tY}f$ has components of the form

$$(4.4) \quad L_{tY}f = \begin{pmatrix} L' & 0 \\ 0 & 0 \end{pmatrix}$$

for any vector field tY in T .

Theorem (4.2). *For both distributions S and T being integrable, the local components of the f_λ -structure are functions independent of the coordinates which are constant along the integral manifold of S in an adapted coordinate system if and only if, for any two vector fields X and Y*

$$(4.5) \quad N(sX, tY) = 0.$$

PROOF. Let us assume that $N(sX, tY) = 0$ for any two vector fields X and Y . Then from theorem (4.1), the tensor field $s(L_{tY}f)s$ vanishes identically for any vector field Y . Hence we have

$$L_{tY}f = 0, \quad \text{and therefore} \quad L' = 0.$$

Which implies that the components of f are independent of the coordinates which are constant along the integral manifold of the distribution S in an adapted coordinate system.

Conversely, if the components of f are independent of these coordinates, then $L' = 0$. Therefore the tensor field $s(L_{tY}f)s$ vanishes identically for any vector field Y . Hence $N(sX, tY) = 0$ for any two vector fields X and Y .

Theorem (4.3). *Suppose that the distributions S and T are both integrable and that an adapted coordinate system has been chosen. Then the components of f are independent of the coordinates which are constant along the integral manifold of S if and only if*

$$(4.6) \quad N(X, Y) = s \cdot N(sX, sY),$$

for any two vector fields X and Y .

PROOF. The proof of the theorem follows by virtue of (3.3) and theorem (4.2).

Definition (4.1) We say that the f_λ -structure is 'integrable' if

(i) the structure f_λ is partially integrable: i.e.,

$$N(sX, sY) = 0;$$

(ii) the distribution T is integrable: i.e.,

$$N(tX, tY) = 0; \quad \text{and}$$

(iii) the components of the f_λ -structure are independent of the coordinates which are constant along the integral manifolds of S in an adapted coordinate system.

Theorem (4.4). *A necessary and sufficient condition for the structure f_λ to be integrable is that*

$$(4.7) \quad N(X, Y) = 0,$$

for any two vector fields X and Y .

PROOF. The proof of the theorem follows from the theorems (3.1), (3.4) and (4.3).

References

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