

On some functional equations in Banach algebras

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Let R denote the field of real numbers, $\mathbf{B}$ a complex Banach algebra, f, g functions mapping R into $\mathbf{B}$. The purpose of this note is to study the solutions of the functional equations

$$(1) \quad f(\xi + \eta) + f(\xi - \eta) = 2f(\xi)g(\eta)$$

(cf. e.g. [1], [7]) and

$$(2) \quad f(\xi + \eta) \cdot f(\xi - \eta) = f(\xi)^2 - f(\eta)^2$$

(see. e.g. [1], [5], [6]) under certain measurability and continuity conditions. We stipulate that φ_e and φ_o denote the even and odd parts, respectively, of the function φ throughout.

Lemma 1. *If f and g satisfy (1) and are strongly measurable, then f is strongly continuous. Let $A_r(H)$ denote the right annihilator of $H = \{f(\xi); \xi \in R\}$, and suppose $A_r(H) = \{0\}$. Then g is continuous, even and satisfies*

$$(3) \quad g(\xi + \eta) + g(\xi - \eta) = 2g(\xi)g(\eta).$$

PROOF. Since (1) can be written as $f(\eta) = 2f\left(\frac{\xi + \eta}{2}\right)g\left(\frac{-\xi + \eta}{2}\right) - f(\xi)$, the first statement follows at once from [2], Theorem 2. From (1) we get $f(\xi)g(\eta) = f(\xi) \cdot g(-\eta)$, which implies $f(\xi)g_o(\eta) = 0$.

Then, by assumption, $g_o(\eta) = 0$ for $\eta \in R$, thus g is even. From (1) we obtain for $\alpha, \xi, \eta \in R$

$$(4) \quad \begin{aligned} f(\alpha)\{g(\xi + \eta) + g(\xi - \eta)\} &= \frac{1}{2}\{f(\alpha + \xi + \eta) + \\ &+ f(\alpha - \xi - \eta) + f(\alpha + \xi - \eta) + f(\alpha - \xi + \eta)\} = \\ &= \{f(\alpha + \xi) + f(\alpha - \xi)\}g(\eta) = 2f(\alpha)g(\xi)g(\eta). \end{aligned}$$

This, by assumption, implies (3) and, by the measurability of g , that g is continuous.

Moreover, we have

Theorem 1. Suppose f, g are continuous and satisfy (1), and g satisfies (3) with $g(0)=j$. Then j is idempotent and there exist uniquely determined elements $a, b, c, d \in B$ such that $a=aj, b=bj=jb, c=cj, jc=0, d=dj$ and

$$(5) \quad f(\xi) = a \left(j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots \right) + \\ + d \left(j\xi + b \frac{\xi^3}{3!} + b^2 \frac{\xi^5}{5!} + \dots \right),$$

$$(6) \quad g(\xi) = \left(j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots \right) + \\ + c \left(j\xi + b \frac{\xi^3}{3!} + b^2 \frac{\xi^5}{5!} + \dots \right).$$

The converse is also true.

PROOF. Since g satisfies (3), the statements concerning g, j, b and c follow from [3]. Moreover, $f(\xi)g_0(\eta)=0$ implies

$$(7) \quad f_e(\xi + \eta) + f_e(\xi - \eta) + f_0(\xi + \eta) + f_0(\xi - \eta) = \\ = 2\{f_e(\xi) + f_0(\xi)\}g_e(\eta)$$

and thus

$$(8) \quad f_e(\xi + \eta) + f_e(\xi - \eta) = 2f_e(\xi)g_e(\eta) = 2f_e(\eta)g_e(\xi).$$

From (8) we get $f_e(\xi) = f_e(\xi)j = f_e(0)g_e(\xi)$, which gives with $f_e(0)=a$ that

$$(9) \quad a = aj \quad \text{and} \quad f_e(\xi) = a \cdot \left(j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots \right).$$

From (7) and (8) we obtain

$$(10) \quad f_0(\xi + \eta) + f_0(\xi - \eta) = 2f_0(\xi)g_e(\eta).$$

Since $jg_e(\eta) = g_e(\eta)j = g_e(\eta)$ for $\eta \in R$, thus $g_e(\eta)$ belongs to the closed subalgebra $B_j = \{x \in B : xj = x = jx\}$ with unit j (cf. e.g. [3]), and for some sufficiently small $\varepsilon > 0$ there exists the inverse r in B_j of $\int_0^\varepsilon g_e(\eta) d\eta$. Integrating (10) we get, by (10) with $\eta=0$,

$$(11) \quad f_0(\xi) = f_0(\xi) \cdot j = \frac{1}{2} \cdot \int_{\xi-\varepsilon}^{\xi+\varepsilon} f_0(\tau) d\tau \cdot r$$

thus f_0 has continuous derivatives of any order on R . Differentiating (10) twice we get $f_0''(\xi + \eta) + f_0''(\xi - \eta) = 2f_0(\xi)g_e''(\eta)$, which gives

$$(12) \quad f_0''(\xi) = f_0(\xi) \cdot b$$

with $f_0(0)=0$ and, in view of (11), $d=f_0'(0)=d \cdot j$. The solution of (12) under these conditions can be obtained as in [3], and is $f_0(\xi) = d \left(j\xi + b \frac{\xi^3}{3!} + b^2 \frac{\xi^5}{5!} + \dots \right)$. Thus we get (5), and since the converse is straightforward, the proof is complete.

Corollary. If f, g are strongly measurable, satisfy (1), and $A_r(H) = \{0\}$, then the conclusions of Theorem 1 hold with $c=0$.

The general measurable solutions of (1) can be somewhat pathological, and in the case of a commutative Banach algebra can be described as follows.

Theorem 2. Suppose $\mathbf{B}$ is a commutative Banach algebra, f, g are strongly measurable and satisfy (1), further N denotes the annihilator of $H = \{f(\xi); \xi \in R\}$. Then there exist uniquely determined elements $j, a, d \in \mathbf{B}$ with $j^2 - j \in N$, $aj = a$, $dj = d$ and $b \in \mathbf{B}$ determined modulo N with $bj - b \in N$ such that

$$(13) \quad f(\xi) = a \left(j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots \right) + d \left(j\xi + b \frac{\xi^3}{3!} + b^2 \frac{\xi^5}{5!} + \dots \right)$$

$$(14) \quad g(\xi) = \left(j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots \right) + r(\xi; b)$$

with $r(0; b) = 0$, where r is a strongly measurable map of R into N . Conversely, if $j, b, a, d \in \mathbf{B}$ with $aj = a$, $dj = d$, N is the annihilator of $\{a\} \cup \{d\}$, r is a map of R into N , and f, g are given by (13) and (14), then f, g satisfy (1).

PROOF. By assumption, N is a closed ideal in $\mathbf{B}$. From (4) we see that $g(\xi + \eta) + g(\xi - \eta) - 2g(\xi)g(\eta) \in N$ for $\xi, \eta \in R$. Let $\mathbf{B}/N$ denote the quotient algebra with norm $\|X\| = \inf \{\|x\|; x \in X\}$ and Q the quotient map of $\mathbf{B}$ onto $\mathbf{B}/N$. Then $\mathbf{B}/N$ is also a commutative Banach algebra and the map $G = Q \circ g$ of R into $\mathbf{B}/N$ is strongly measurable and satisfies (3). According to [3] there exist uniquely determined elements $J, B \in \mathbf{B}/N$ such that $G(\xi) = J + B \frac{\xi^2}{2!} + B^2 \frac{\xi^4}{4!} + \dots$ with $J^2 = J$, $BJ = B$. If we put $g(0) = j \in J$, and choose an element $b \in B$, we have $j^2 - j \in N$, $bj - b \in N$ and that $r(\xi; b) = g(\xi) - \left(j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots \right) \in N$, where r is strongly measurable with $r(0; b) = 0$. Introducing the notation $g_d(\xi) = j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots$, we get from (1) as in the proof of Theorem 1 $f_e(\xi) = f(0)g_d(\xi) = f_e(\xi)j$, which gives with $f(0) = a$, $a = aj$ and $f_e(\xi) = a \left(j + b \frac{\xi^2}{2!} + b^2 \frac{\xi^4}{4!} + \dots \right)$. Similarly, we obtain

$$(15) \quad f_0(\xi + \eta) + f_0(\xi - \eta) = 2f_0(\xi)g_d(\eta)$$

and hence

$$(16) \quad \frac{1}{2} \cdot \int_{\xi-\varepsilon}^{\xi+\varepsilon} f_0(\tau) d\tau = f_0(\xi) \cdot \int_0^\varepsilon g_d(\eta) d\eta \quad (\varepsilon > 0).$$

With the notation $x(\varepsilon) = \frac{1}{\varepsilon} \int_0^\varepsilon g_d(\eta) d\eta$ we have $\lim_{\varepsilon \rightarrow 0} x(\varepsilon) = j$, thus with $X(\varepsilon) = Qx(\varepsilon)$ we get $\lim_{\varepsilon \rightarrow 0} X(\varepsilon) = J$. If $\mathbf{B}_J = \{X \in \mathbf{B}/N : XJ = X\}$, then $\mathbf{B}_J$ is a closed commutative subalgebra of $\mathbf{B}/N$ with unit J . Since $g_d(\eta) \cdot j = j + m + (b + n) \frac{\eta^2}{2!} + b(b + n) \frac{\eta^4}{4!} +$

$+b^2(b+n)\frac{\eta^6}{6!}+\dots=g_d(\eta)+p(\eta)$, where $m=j^2-j\in N$, $n=bj-b\in N$, $p(\eta)\in N$ for $\eta\in R$, we have $x(\varepsilon)\cdot j=\frac{1}{\varepsilon}\int_0^\varepsilon g_d(\eta)\cdot j\,d\eta=x(\varepsilon)+\frac{1}{\varepsilon}\int_0^\varepsilon p(\eta)\,d\eta\in X(\varepsilon)$, for N is closed. Thus $X(\varepsilon)\in\mathbf{B}_J$ for every $\varepsilon>0$, and for some $\varepsilon>0$ there exists a $Y(\varepsilon)\in\mathbf{B}_J$ with $X(\varepsilon)Y(\varepsilon)=J$, consequently there exists a $y(\varepsilon)\in\mathbf{B}$ with $x(\varepsilon)y(\varepsilon)-j=q\in N$. It follows that $\int_0^\varepsilon g_d(\eta)\,d\eta\cdot\frac{1}{\varepsilon}y(\varepsilon)=j+q$, and (15) and the definition of N give $f_0(\xi)\cdot(j+q)=f_0(\xi)$. Thus (16) implies $f_0(\xi)=\frac{1}{2\varepsilon}y(\varepsilon)\int_{\xi-\varepsilon}^{\xi+\varepsilon}f_0(\tau)\,d\tau$, consequently f_0 has derivatives of every order on R . A similar argument as in the proof of Theorem 1 yields that with $d=f'_0(0)=f'_0(0)j=dj$ we have

$$f_0(\xi)=d\left(j\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\dots\right),$$

while the converse part of the theorem can be proved by straightforward calculation.

In what follows we deal with the equation (2). We always assume that $\mathbf{B}$ is an algebra with unit e and, to avoid pathological phenomena (cf. [5]) that $f(\alpha)$ is regular for some $\alpha\in R$.

Lemma 2. *Suppose $\mathbf{B}$ is an algebra with unit e , $f:R\rightarrow\mathbf{B}$ satisfies (2) and for some $\alpha\in R$ $f(\alpha)^{-1}$ exists. Then f is odd, $g(\xi)=\frac{1}{2}f(\alpha)^{-1}\{f(\xi+\alpha)-f(\xi-\alpha)\}$ satisfies (3) with $g(0)=e$, and f, g satisfy (1).*

PROOF. If we put (η, ξ) instead of (ξ, η) in (2), we get $f(\alpha)\{f(\xi)+f(-\xi)\}=0$ for $\xi\in R$. Thus f is odd, and this implies that $f(\xi), f(\eta)$ commute for $\xi, \eta\in R$ (cf. [5]). Consequently, $f(\alpha)^{-1}$ also commutes with $f(\xi)$ for every $\xi\in R$, and the calculation in [1], p. 137 applies and gives that g satisfies (3) with $g(0)=e$. Moreover, a calculation similar to that in [1], p. 138 yields that f, g satisfy (1), and the lemma is proved.

The following theorem generalizes some results of S. KUREPA (cf. [5]).

Theorem 3. *Suppose $\mathbf{B}$ is a Banach algebra with unit e , $f:R\rightarrow\mathbf{B}$ satisfies (2) and is strongly measurable on a set $P\subset R$ of positive Lebesgue measure, further $f(\alpha)$ is regular for some $\alpha\in R$. Then there exist uniquely determined elements $b, d\in\mathbf{B}$ with $bd=db$ such that $f(\xi)=d\left\{e\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\dots\right\}$. Conversely, if f is of the above form with $bd=db$, then f satisfies (2).*

PROOF. We first show that under these conditions 0 is in the closure of the set $H=\{\xi\in R:f(\xi)\text{ regular}\}$. Indeed, according to Lemma 2 we have for every $\xi\in R$ $f(2\xi)=2f(\xi)g(\xi)$, where $f(\xi)$ and $g(\xi)$ commute. If we assume that $f(\xi)$ is singular for $|\xi|<\varepsilon$ with $\varepsilon>0$, then it follows that $f(\xi)$ is singular also for $|\xi|<2\varepsilon$ and thus for every $\xi\in R$, which contradicts the assumptions of the theorem.

Moreover, according to [4] the map $\varphi:R^2\rightarrow[0, \infty)$, $\varphi(\xi_1, \xi_2)=m\{(P+\xi_1)\cap(P+\xi_2)\}$ (m denotes Lebesgue measure) is continuous with $\varphi(0, 0)=m(P)>0$.

By the above reasoning, for some $\alpha \in H$ we have $m(P_\alpha) = m\{(P+\alpha) \cap (P-\alpha)\} > 0$. $\xi \in P_\alpha$ implies $\xi + \alpha, \xi - \alpha \in P$, consequently $g(\xi) = \frac{1}{2} f(\alpha)^{-1} \{f(\xi + \alpha) - f(\xi - \alpha)\}$ is measurable on P_α and [3], Corollary to Prop. 2. ensures the continuity of g . Moreover, there exists a compact $K \subset P$ such that $m(K) > 0$, and the restriction of f to K is uniformly continuous (cf. e.g. [2]), while the restriction of g to K is also uniformly continuous.

Suppose $\varepsilon > 0$ and find a $\delta > 0$ such that $|\eta| < \delta$ implies $L_\eta = K \cap (K + \eta) \cap (K - \eta) \neq \emptyset$ and that $\|f(\xi) - f(\xi + \eta)\| < \varepsilon$ and $\|g(\xi) - g(\xi + \eta)\| < \varepsilon$ whenever $\xi, \xi + \eta \in K$. Since f, g satisfy (1), for $\xi, \eta \in R$ we have $f(2\eta) = 2f(\xi + \eta)g(\xi - \eta) - f(2\xi)$ and, applying again (1)

$$(17) \quad f(2\eta) = 2f(\xi + \eta)g(\xi - \eta) - 2f(\xi)g(\xi) \quad (\xi, \eta \in R).$$

If $|\eta| < \delta$, then there exists a $\xi \in L_\eta$, for which we have $\xi, \xi - \eta, \xi + \eta \in K$ and, putting $S = \max \{\|f(\xi)\|, \|g(\xi)\| : \xi \in K\}$ we obtain by (17)

$$\begin{aligned} \frac{1}{2} \|f(2\eta)\| &\leq \|f(\xi + \eta)\| \cdot \|g(\xi - \eta) - g(\xi)\| + \\ &\quad + \|f(\xi + \eta) - f(\xi)\| \cdot \|g(\xi)\| \leq 2S\varepsilon, \end{aligned}$$

thus f is continuous at 0. Since f is odd, we get

$$\begin{aligned} f(\xi + \eta) - f(\xi) &= f(\xi + \eta) + f(-\xi) = \\ &= 2f(\eta/2)g(\xi + \eta/2) \quad (\xi, \eta \in R) \end{aligned}$$

and the continuity of g implies the continuity of f on R .

Now we apply Theorem 1 and establish by (5) that $f(\xi) = d \left(e\xi + b \frac{\xi^3}{3!} + b^2 \frac{\xi^5}{5!} + \dots \right)$ with $d = f'(0)$. Moreover, since $f(\xi)$ and $g(\eta)$ commute for $\xi, \eta \in R$ we also obtain $f'(\xi) \cdot g''(\eta) = g''(\eta) \cdot f'(\xi)$ and, putting $\xi = \eta = 0$, $db = bd$. The converse can be obtained by direct calculation, thus the theorem is proved.

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