Research problems in number theory

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In this paper *I* state some problems in number theory that *I* was interested in the last years. One part of them was published in [1] written in hungarian.

I. Characterization of additive functions

Let $f(n)$ be a completely additive function, i.e. suppose that the relation

$$f(mn) = f(m)+f(n)$$

holds for every m, n.

I raised the following question [2], which we state now as

Conjecture 1. If $f(p+1)=0$ for every prime p, then $f(n)=0$ identically.

In the same paper I have proved a little more week assertion, on the assumption of the Riemann—Piltz conjecture namely

Theorem A. *If* $f(p+1)=0$ *for every prime* p, *and* $f(p)=0$ *for every prime* p *less than* $K-K$ *being an effective numerical constant —, then* $f(n)=0$ *identically.*

In [3] *I* proved this assertion with ineffective K, without any condition. The proof is based on the large sieve' theorem due to A. I. VINOGRADOV and E. BOMBIERI. The ineffectivity of K has a (near) connection with the problem of effectivization concerning the Siegel-roots of the L-functions. Presently ELLIOTT proved conjecture 1. (See [4].)

Conjecture 2. If

$$f(p+1) \geqq f(p'+1) \quad (p > p')$$

for every pair of primes p, p', then $f(n)$ is a constant multiple of $\log n$.

This would be a generalization of the well known theorem due to P. ERDŐS states that a monotonic $f(n)$ must be a constant multiple of $\log n$. Furthermore, if Conjecture 2 is true, then Conjecture 1 is true too. If the equation $ap-bp'=1$ has an infinity of solutions in primes p, p', for every relative prime pairs a, b of integers, then Conjecture 2 is true.

Presently we can prove only the following assertion: *if* $f(P^3+1)$ *is monotonic, where* P_3 *runs over the numbers having at most three prime divisors, then* $f(n)$ *is a constant multiple of* $\log n$. *The equation*

$$nP_3-(n+1)P_3' = -1$$

has an infinity of solutions — as we can see by the Selberg's sieve — hence

$$n(P_3+1) = (n+1)(P_3'+1),$$

and so

$$f(n) \leqq f(n+1).$$

But a monotonic additive function must be a constant multiple of $\log n$, as proved P. Erdős.

Conjecture 4. If $f(p+1)\geqq 0$ for every prime p, then $f(n)\geqq 0$ for every natural number n.

This would be a consequence of the following hopeless conjecture: for every natural number a, there exists a natural number K, such that the equation

$$p+1 = Ka^n$$

has an infinity of solutions, when $n=1, 2, \ldots,$ and p runs through the primes.

Conjecture 3. Suppose that

$$|f(p+1)| \leqq C \log (p+1) \tag{1.1}$$

for every prime p. Then

$$|f(n)| \leqq AC \log n, \tag{1.2}$$

where A is a suitable absolute constant.

I have proved [5] the following conditional result.

Theorem B. *Assuming the Riemann—Piltz conjecture, from* (1.1) *it follows that*

$$|f(n)| \leqq K_f (\log n) \cdot \log\log 10n$$

where K_f *a constant, which depends on* f.

Now *I* can prove an unconditional result, but this is very wrong.

Presently Elliott proved Conjecture 4*) and the following Conjecture 5, too.

Let now $f(n)$ denote an arbitrary additive function.

Conjecture 5. Let

$$E(x) = \max_{p\leqq x} |f(p+1)|, \quad M(x) = \max_{n\leqq x} |f(n)|.$$

Then there exist absolute numerical constants A, B such that

$$M(x) \leqq AE(x^B).$$

Perhaps this assertion holds for $B=1$.

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*) P. D. T. A. Elliot, On two conjoctures of Kátai, *Acta Arithm.* **30** (1976), 341—375.

Conjenture 6. The following inequalities hold:

$$x^{-1}\sum_{n\leqq x}|f(n)| \leqq Ax^{-B}(\log x^B)\sum_{p\leqq x^B}|f(p+1)|,$$

$$\frac{\log x}{x}\sum_{p\leqq x}|f(p+1)| \leqq Ax^{-B}\sum_{n\leqq x^B}|f(n)|,$$

where A and B are suitable absolute constans.

Conjecture 7. The following inequality is true:

$$\max_{2n\leqq x}|f(2n)-f(2n-2)| \leqq A\max_{p\leqq x^B}|f(p+1)-f(p-1)|.$$

Conjecture 8. The following inequalities hold

$$x^{-1}\log x\sum_{p\leqq x}|f(p+1)-f(p-1)| \leqq$$

$$\leqq Ax^{-B}\sum_{2n\leqq x^B}|f(2n)-f(2n-2)|,$$

$$x^{-1}\sum_{2n\leqq x}|f(2n)-f(2n-2)| \leqq Ax^B(\log x^B)\sum_{p\leqq x^B}|f(p+1)-f(p-1)|,$$

where A and B are suitable absolute constants.

Erdős proved that from $f(n+1)-f(n)\to 0$ $(n\to\infty)$ it follows that $f(n)=c\log n$. Generalizing this, *I* proved that from

$$\underline{\lim}_{n\to\infty}\Delta^k f(n) \geqq 0$$

it follows that $f(n)=c\log n$ [6]. The same assertion for $k=1$ was stated without proof by P. ERDŐS [7], and have been proved by I. KÁTAI [8], A. MÁTÉ [9], too.

I proved that when $f(n)$ and $g(n)$ are additive functions and

$$g(n+1)-f(n)\to 0 \quad (n\to\infty),$$

then $f(n)=g(n)=c\log n$.

I think holds the following assertion, which we state as

Conjecture 9. Let $a_i, b_i\,(i=1,\ldots,k)$ be distinct pairs of natural numbers, $a_i>0$ $(i=1,\ldots,k)$, $f_i(n)$ be additive functions, such that

$$\sum_{i=1}^{k}c_i f_i(a_i\cdot n+b_i)\to 0 \quad (n\to\infty),$$

$c_1,\ldots,c_k$ arbitrary constants. Then $f_i(n)=d_i\log n+l_i(n)$, where $l_i(n)$ must be of finite supports, i.e.

$$l_i(p^z)=0$$

except at most for a finite set of prime powers p^z. Furthermore

$$\sum_{i=1}^{k}c_i l_i(a_i n+b_i)=0 \quad (n=1,2,\ldots).$$

Wirsing [10] and *I* [11] proved independently that from

$$\frac{1}{x}\sum_{n\leqq x}|f(n+1)-f(n)|\to 0 \tag{1.3}$$

it follows that $f(n)=c\log n$, which was an old conjecture of P. Erdős

Later WIRSING deduced this assertion assuming only that

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{x\leqq n\leqq(1+\gamma)x}|f(n+1)-f(n)|=0,\quad \gamma>0,\quad \text{constant.}$$

WIRSING has proved another old conjecture of P. Erdős, namely that from

$$|f(n+1)-f(n)|\leqq K(=\text{constant})$$

it follows that

$$f(n)=c\log n+g(n),$$

$g(n)$ being a bounded (additive) function.

I stated in [12], that from

$$|g(n+1)-f(n)|\leqq K$$

(f and g are arbitrary additive functions) it follows that

$$f(n)=c\log n+h_1(n),\quad g(n)=c\log n+h_2(n),$$

$h_1(n)$, $h_2(n)$ are bounded functions. This assertion was proved recently by J. MAUCLAIRE*)

Conjecture 10. If for an additive function f the values $f(p+1)$ are integers for each primes p, then $f(n)=$ integer for all natural number n.

II. Distribution of additive functions

We say that the additive function $f(n)$ has a limit-distribution on the set of "prime plus one"-s, when the frequency

$$\frac{1}{\pi(N)}N\{p\leqq N,\ f(p+1)<x\}$$

tends to a distribution function $F(x)$ at all the continuity points of its. It was proved in [13] the following analogon of Erdős—Wintner-theorem. If the series

a) $$\sum_{|f(p)|\leqq 1}\frac{f(p)}{p}$$

b) $$\sum_{|f(p)|\leqq 1}\frac{f^2(p)}{p}$$

c) $$\sum_{|f(p)|>1}1/p$$

*) J.-L. MAUCLAIRE, On a problem of Kátai, *Acta. Sci. Math.* **36** (1974), 205—207.

converge, then $f(p+1)$ has a limit-distribution. The question of the necessity of these conditions was asked by Erdős and Kubilius.

Conjecture 11. If $f(p+1)$ has a limit-distribution, then the series' a., b., c. are convergent.

I proved this only on the additional condition $|f(q)|$ is bounded on the set of primes. Recently D. Elliott proved this for non-negative $f(n)$.

Let A be an infinite sequence of natural numbers. We say that an additive functions $f(n)$ defined at least on A has a limit distribution $F(x)$, when

$$\lim_{N\to\infty} \frac{1}{A(N)} N\{a\in A,\ f(a) < x\} = F(x),$$

at every continuity points of $F(x)$. Here

$$A(N) = \sum_{\substack{a\in A\\ a\leqq A}} 1.$$

We say that a sequence Q of prime-powers is an "unessential sequence" for the sequence A, if the existence of the limit distribution of $f(n)$ on A, does not depend on the values' $f(n)$ taken on Q. In other words Q is an "unessential sequence", if the following statement is valid: if there exists a limit-distribution for $f(n)$, then it exists too, when we change the values $f(q)$ for $q\in Q$, arbitrarily.

From the Erdős—Wintner theorem we can easily deduce the following assertion. If A is the sequence of all natural numbers, then Q is an unessential sequence if and only if

$$\sum_{q\in Q} 1/q < \infty. \tag{2.1}$$

From the previous conjecture it would be follow

Conjecture 12. Q is an unessential sequence for the sequence "prime plus one" s, if (2.1) holds.

Let $q(n)$ be an irreducible polynomial of degree k ($\geqq 2$). Let $A=\{g(n), n=1, 2, \ldots\}$.

From a result of P. Erdős we can deduce easily that (2.1) is not enough for the Q to be unessential.

HOOLEY proved the following assertion for $k\geqq 3$. The number of those $n\leqq x$, for which there exists at least one prime $p>y(x)$, $p^{k-2}|g(n)$ is $o(x)$, when $y(x)\to\infty$ arbitrarily slowly.

Conjecture 13. Let $Q=\{p^\alpha, \alpha\geqq 2\}$. Then Q is an unessential sequence for $A=\{g(n); n=1, 2, \ldots\}$.

Conjecture 14. Let $g(n)$ be an irreducible polynomial, $A=\{g(p), p \text{ prime}\}$. Then $Q=\{p^\alpha, \alpha\geqq 2\}$ is an unessential sequence for A.

III. Iteration of multiplicative functions

Let $d(n)$ denote the number of divisors of n, and $d_k(n)=d(d_{k-1}(n))$, $(d_1(n)=d(n))$ the k'th iterative of its. Similarly, let $\log_k n$ denote the k'th iterative of $\log n$.

Erdős and I proved [14] that

$$d_k(n) < \exp[(\log n)^{1/l_k+\varepsilon}],$$

for every sufficiently large n, and

$$d_k(n) > \exp[(\log n)^{1/l_k-\varepsilon}],$$

infinitely many times. Here

$$l_1 = 1, \quad l_2 = 2, \quad l_k = l_{k-2}+l_{k-1}.$$

Let $K(n)$ denote the smallest k for which $d_k(n)=2$. We think that the following assertion is true

Conjecture 15. The relation

$$x^{-1}\sum_{n\leqq x} K(n) = (1+o(1))\,L(x)$$

holds, where $L(x)$ is defined as the integer satisfying

$$1 \leqq \log_{L(x)} x < e.$$

Let

$$D_k(x) = \sum_{n\leqq x} d_k(n).$$

Bellman and Shapiro stated the hypothesis

$$D_k(x) = c_k(1+o(1))\,x\log_k x \quad (c_k > 0)$$

for every k. We proved this for $k=2, 3, 4$. (see e.g. [15])

For $k=5$, I can not prove that

$$(0 <)\, A_1 < \frac{D_5(x)}{x\log_5 x} < A_2 (< \infty).$$

Let $g(n)=d_{K(n)-1}(n)$. It is evident that $g(n)$ is a prime-number for $n\geqq 2$.

Conjecture 16. $g(n)=3$ for almost all n.

Let $U(n)$ denote the number of distinct prime divisors of n, and $V(n)$ the all prime divisors of n; i.e. for $n=p_1^{\alpha_1}\ldots p_r^{\alpha_r}$ let

$$U(n) = r, \quad V(n) = \alpha_1+\ldots+\alpha_r.$$

Lez $U_k(n)$, $V_k(n)$ denote the iterative functions. We state $U(0)=V(0)=0$.

Let

$$\mu_k(n) = \begin{cases} 1 & \text{if } U_j(n) = V_j(n) \text{ for } j \leqq k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_T(n) = \begin{cases} 1 & \text{if } \mu_k(n) = 1 \text{ for every } k \\ 0 & \text{otherwise,} \end{cases}$$

$$M_k(x) = \sum_{n \leqq x} \mu_k(n); \quad M_T(x) = \sum_{n \leqq x} \mu_T(n).$$

Conjecture 16. The relation

$$M_k(x) = (1 + o(1)) \left(\frac{6}{\pi^2}\right)^k x$$

holds for every $k \geqq 1$.

For $k=1$ it is well known, *I* can prove this for $k=2, 3$. The case $k=4$ seems to be hard.

Conjecture 17. The relation

$$\log x^{-1} M_T(x) = (1 + o(1)) L(x) \log \frac{6}{\pi^2}$$

holds.

IV. Local behaviour of number-theoretical functions

4.1. Let $f(n)$ be a completely multiplicative function, i.e. $f(mn) = f(m) \cdot f(n)$ for every pairs of natural numbers. Assume that $f(n)$ does not take on the value zero and that $f(n) \not\equiv 1$. *I* have proved that for $n \geqq n_0 (= n_0(f))$ the relation

$$f(n) = f(n+1) = \ldots = f(n+j) \quad j = 4[\sqrt{n}]$$

does not hold [16]. The elementary method which was used allows to change the constant 4 to $2+\varepsilon$.

Especially hence it follows, that a completely multiplicative function having only two vales — $+1$, -1 — takes the both values in every interval $n, n+(2+\varepsilon)\sqrt{n}$ for all n large enough.

Although the method is very simple, as *I* know, there no exist better result for the Liouville function, too.

Conjecture 18. If $f(n)$ is a completely multiplicative function, takes no the value zero, and $f(n) \not\equiv 1$, then it takes on at least two distinct values in every interval $[n, n+n^{\varepsilon}]$, whenever $n > n_0(f, \varepsilon)$, for every positive constant ε.

Perhaps the following more general assertion is valid too.

Conjecture 19. Let $f(n)$ be a completely multiplicative function, that takes no zeros on, and have at least k distinct values. Then it takes on at least k distinct values in every interval $[n, n+n^{\varepsilon}]$, whenever $n > n_0(\varepsilon, f)$, ε being an arbitrary positive constant.

Let $f(n)$ be a multiplicative function defined on the set of square-free integers. I have proved [17] that every $f(n)$ which is not identically equal to 1, and never

zero takes on at least two values in the interval $N, N+N^{\vartheta}$ $\vartheta=0,62$, whenever $N>N_0(f, \varepsilon)$.

Perhaps the following assertion holds too.

Conjecture 20. If $f(n)$ is a multiplicative function, never zero, and takes on at least two values, then $f(n)$ takes on at least two values in every interval $[N, N+c\sqrt{N}]$, if $N>N_0(f)$. c being a suitable absolute positive constant.

Let $\lambda(n)$ denote the Liouville-function, and $m(n)$ the least positive k, such that $\lambda(n+k)=-\lambda(n)$. From the cited theorem it follows that $m(n)<4\sqrt{n}$ for $n>n_0$.

Conjecture 21. The relation

$$m(n) = 0(\log n)$$

holds.

On the opposite side *I* could not prove presently that $\overline{\lim_n} m(n)\geqq 3$.

Conjecture 22. The relation

$$\overline{\lim_{n\to\infty}} m(n) = \infty.$$

holds.

4.2. For additive functions we can prove similar results.

IVÁNYI and *I* proved the following assertion. If $f(n)$ is a completely additive function, $N_1<N_2<\ldots$ an infinite sequence of integers, $\varepsilon>0$ an arbitrary positive constant, such that

$$f(n) \leqq f(n+1) \quad \text{when} \quad n\in[N_j,\ N_j+(2+\varepsilon)\sqrt{N_j}]$$

$j=1, 2, \ldots$ then $f(n)$ is a constant multiple of $\log n$. [18].

We think that the following assertion holds.

Conjecture 23. If $f(n)$ is an additive function such that

$$f(n) \leqq f(n+1) \quad n\in[N_j, N_j+N_j^{\varepsilon}],$$

for an infinite sequence $N_1<N_2<\ldots$, and ε is an arbitrary positive constant, then $f(n)$ is a constant multiple of $\log n$.

Let $f(n)$ denote a number-theoretical function having only two values $+1, -1$. Let

$$N_f(x;\ \varepsilon_0, \cdots, \varepsilon_k) = N\{n \leqq x;\ f(n+i)\} = \varepsilon_i, \quad i = 0, \ldots, k.$$

$$\varepsilon_i = \pm 1, \quad i = 0, \ldots, k$$

We say that the function $f(n)$ is of a normal type, if

$$x^{-1} N_f(x;\ \varepsilon_0, \ldots, \varepsilon_k) \to 2^{-k-1} \quad (x \to \infty)$$

for every k, and every choice of the values $\varepsilon_i=\pm 1$, $(i=0, \ldots, k)$.

Now we define a metrization on the set of completely multiplicative functions which take the values $+1$ and -1 only.

Let p_n denote the n-th prime number. Let (Ω, A, P) be a probability space and $\xi_n=\xi_n(\omega)$ $(n=1, 2, \ldots)$ be a sequence of independent random variables with the distribution $P(\xi_n=+1)=P(\xi_n=-1)=1/2$. Let $f(n;\omega)$ denote the completely multiplicative function which we define on the set of primes by $f(p_n;\omega)=\xi_n(\omega)$. We have proved that for almost all $\omega, f(n,\omega)$ is a function of normal type [19].

Conjecture 24. The Liouville-function $\lambda(n)$ is of normal-type.

We could not give a construction for a multiplicative function of normal type.

Conjecture 25. Let $f(n)$ be a completely multiplicative function which takes the values $+1$ and -1 only. If

$$\sum_{f(p)=-1} 1/p = \infty, \tag{4.1}$$

then

$$\frac{1}{x} N_f(x, \varepsilon_1, \varepsilon_2) \to 1/4$$

for every choice of $\varepsilon_1=\pm 1, \varepsilon_2=\pm 1$.

This would be a generalization of a result due to WIRSING [20].

Presently *I* can prove only, that from (4.1), the inequalities

$$\liminf_{x\to\infty} \frac{1}{x} N_f(x, 1, 1) \geqq 1/12$$

$$\liminf_{x\to\infty} \frac{1}{x} N_f(x, -1, -1) \geqq 1/12$$

follows. For $\varepsilon_1 \neq \varepsilon_2$ *I* can prove only the infinity of $n's$ satisfying $f(n)=\varepsilon_1 f(n+1)=\varepsilon_2$.

Let $f(n)$ $n=1, 2, \ldots$ be an arbitrary sequence of the values $+1, -1$, and take

$$h_f(n) = \sum_{v=1}^{n-1} f(v)\cdot f(n-v).$$

Corrádi and *I* proved easily that

$$\overline{\lim_{n\to\infty}} \frac{|h_f(n)|}{\sqrt{n}} \geqq 1 \quad ([21]).$$

Furthermore we believe that the following assertion holds.

Conjecture 26.

$$\overline{\lim_{n\to\infty}} \frac{|h_f(n)|}{\sqrt{n}} = \infty$$

for every $f(n)$.

I think the following assertion is valid.

Conjecture 27. If $f(n)$ is a completely multiplicative function, and (4.1) is divergent, then

$$\underline{\lim_{n\to\infty}} h_f(n) = -\infty.$$

Let

$$h(n) = \sum_{d|n} l(d)\, l\left(\frac{n}{d}\right),$$

where $l(m)$ is an arbitrary number-theoretical function which takes the values $+1, -1$ only. *I* have proved the inequality

$$\sum_{n=1}^{\infty} \frac{|h(n)|^2}{n^\sigma} \geqq \sum_{n=1} \frac{\tau(n)}{n^\sigma} \quad (\sigma > 1),$$

where $\tau(n)$ is the number of divisors of n. Hence we can deduce easily that

$$\overline{\lim_{n\to\infty}} \frac{|h(n)|}{\sqrt{\log n}} \geqq 1.$$

Conjecture 28. The relation

$$\overline{\lim_{n\to\infty}} \frac{|h(n)|}{\sqrt{\tau(n)+\log n}} = \infty$$

holds.

V. The values of multiplicative functions on special sequences

Chowla stated the following conjecture. If $g(x)$ is a polynomial with integer coefficient and it can not be represented as $ch^2(x)$, then $\lambda(g(n))=1$, and $\lambda(g(n))=-1$, for infinitely many n. λ denotes the Liouville function.

To prove that the sequence has an infinity of sign-changes is very simple. The Pell-equation $n^2+1=2(m^2+1)$ has an infinity of solutions (n, m), and for these values $\lambda(n^2+1)=-\lambda(m^2+1)$. After CHOWLA *I* think that the following conjecture is true.

Conjecture 29. If a is an arbitrary positive integer, then the sequence $\lambda(n^2+a)$ $n=1, 2, \ldots$ takes both of the values $+1, -1$ infinitely many times.

Perhaps the following assertion is true.

Conjecture 30. If a is an arbitrary positive integer, then

$$\frac{1}{x} \sum_{n\leqq x} \lambda(n^2+a) \to 0 \quad (x \to \infty).$$

Presently *I* do not prove that

$$\liminf \frac{1}{x} \left| \sum_{n\leqq x} \lambda(n^2+1) \right| < 1.$$

Conjecture 31. Let ε be an arbitrary positive number. Then $\lambda(n^2+1)$ takes both of the values $+1, -1$ in the interval $[x, (1+\varepsilon)x]$, if x is large enough.

It would be interesting to get results for the values of λ on the set of shifted primes.

Conjecture 32. The sequence $\lambda(p+1)$ takes the values $+1, -1$ infinitely many times, when p runs over the primes.

Let $l_+(n)$, and $l_-(n)$ denote the number of those primes $p<n$, for which $\lambda(n-p)=1$, $\lambda(n-p)=-1$, respectively.

Conjecture 33. The relation

$$l(n) \stackrel{\text{def}}{=} \min\left(l_+(n), \quad l_-(n)\right) \to \infty \quad (n \to \infty)$$

holds.

Presently I can not prove the more weak assertion $\lambda(n)>0$ for n large enough.

It is well-known that

$$\sum_{n \leqq x} \lambda(kn+l) = o(x)$$

for every k and l, and hence it follows that the function $\lambda(n)$ takes both of the possible values in every arithmetical progression.

Let $L(k)$ denote the smallest integer x so that for every $l=0, 1, \ldots, k-1$ in the arithmetical progression $kn+l \leqq x$ the function $\lambda(n)$ takes both of its possible values. Using the theorem of Linnik on the smallest prime in arithmetical progression we get immediately that

$$L(k) < k^c,$$

c being a constant.

Conjecture 34. The relation

$$L(k) < ck \log k$$

holds.

If k is a prime number then by the result of Burgess, and of Linnik — A. I. Vinogradov on the least quadratic non residue and on the least prime quadratic residue we can deduce easily that

$$L(k) < k^{2+1/4}$$

On the opposite side I think that the following assertion is true.

Conjecture 35. The relation

$$\overline{\lim_{k \to \infty}} \frac{L(k)}{k} = \infty$$

is true.

This would be a consequence of the assertion that the sequence of prime numbers have an arithmetical progression of arbitrary length. However the last conjecture seems to be easier.

Similar questions can be raised for the Moebius-function, and other functions, instead of $\lambda(n)$.

V. On the additive representation of natural numbers

5.1. Let $A=\{1=a_1, a_2, \ldots\}$ be an infinite sequence of integers. Every natural number can be written in the form

$$n = a_{i_1}+a_{i_2}+\ldots+a_{i_v}, \tag{5.1}$$

where a_{i_1} is the greatest element of A not greater than n, and in general a_{i_k} is the greatest element of A not greater than $n-(a_{i_1}+\ldots+a_{i_{k-1}})$. This representation is unique.

Let us denote by $\alpha(n)$ the length of this representation, i.e. for n in (5.1) let $\alpha(n)=v$.

Let

$$B_k(x) = \sum_{n\leqq x} \alpha^k(n), \quad \varrho_d(x) = \sum_{\substack{i \\ a_{i+1}-a_i=d \\ a_{i+1}\leqq x}} 1$$

I proved [22] that on the assumption of the existence of the limits $\lim_x x^{-1}\varrho_d(x)=\varrho_d$, and $\Sigma d\,\varrho_d=1$, there exist the limits

$$\lim_{x\to\infty} x^{-1}B_k(x)$$

for every natural number k, furthermore the limits

$$\lim_x x^{-1}N\{e\leqq x;\ \alpha(n)=k\} = c_k$$

there exist for every k, and $\Sigma c_k=1$.

The assumptions are satisfied for example for the set of square-free numbers.

It is evident that $\alpha(n)$ is bounded if and only if $a_{i+1}-a_i$ is a bounded sequence. The question of the connection of the maximal order of $\alpha(n)$ and of $a_{i+1}-a_i$ is open.

5.2. *I* could not give a good estimation for the maximal order of $\alpha(n)$, when A is the sequence of square free numbers. Perhaps the following assertion is true.

Conjecture 36. If A is the set of square-free numbers, then

$$\frac{\alpha(n)}{\log\log n} \to 0 \quad (n\to\infty).$$

The assertion $\alpha(n)<c\log\log n$ is a straightforward consequence of the fact $a_{i+1}-a_i<Ca_i^{\delta}$ with $\delta<1$.

5.3. *I* considered the case, when A is the set of square numbers, and proved that

$$\sum_{n\leqq x} |\alpha(n)-\overset{2}{\log}\overset{2}{\log} x|^k = O(x)$$

for every fixed k, and furthermore that

$$\alpha(n) < \overset{2}{\log}\overset{2}{\log} n+5.$$

Erdős raised the question whether there exists a constant c such that $\alpha(n) > \overset{2}{\log}\overset{2}{\log} n - c$ for all but $o(x)$ of $n \leqq x$, or not. This question is open presently. An other question due to Erdős that $\alpha(n) - [\overset{2}{\log}\overset{2}{\log} n]$ have a limit distribution in some sense, or not.

5.4. Let now A be the set of the prime numbers and $1 : A = \{1 < p_1 < p_2 < \ldots\}$. From the theorem of Hoheisel we get immediately that

$$\alpha(n) < c \log\log n.$$

The following conjecture seems to be hard.

Conjecture 37. The relation

$$\frac{\alpha(n)}{\log\log n} \to 0$$

holds.

Conjecture 38. The relation

(5.2) a $$\limsup_{n\to\infty} \frac{\alpha(n)}{L(n)} \leqq 1$$

holds, where $L(x)$ denotes the integer for which $1 \leqq \log_{L(x)} x < e$.

This would be a consequence of the Cramer's hypothesis stating that $p_{n+1} - p_n \ll \ll (\log p_n)^2$.

It is easy to prove that the constant 1 cannot be substitute by a smaller one in the right hand side of (5.2). It is an immediate consequence of the inequality

$$\underline{\lim}_{x\to\infty} B_k(x) / x L^k(x) \geqq 1,$$

which we can deduce easily by using the Brun's sieve.

Conjecture 39. For every k the relation

$$B_k(x) = \big(1 + o(1)\big) x L^k(x) \quad (x \to \infty)$$

holds.

I can deduce this relation from the Riemann hypothesis [23]. Hence it would be follow the assertion which we state as

Conjecture 40. Leaving a set of zero-density of integers n, on the remainder set the relation

$$\lim_{n\to\infty} \frac{\alpha(n)}{L(n)} = 1,$$

holds.

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(Received January 1, 1975.)