

Some remarks on valuations and subfields of given codimension in algebraically closed fields

To the memory of Andor Kertész

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If K is an algebraically closed field and L a subfield for which $[K:L] < \infty$ the Artin-Schreier theorem says that $[K:L] = 1$ or 2 . When $[K:L] = 2$ L is a real closed ("maximal ordered") subfield of K . KNOPFMACHER and SINCLAIR [4] asked if a subfield L of the complex number field $\mathbb{C}$ such that $[\mathbb{C}:L] = 2$ would be isomorphic to the real number field $\mathbb{R}$. BIALYNICKI-BIRULA [1] gave an example of such an L where $L \not\cong \mathbb{R}$, and subsequently studied subfields of prescribed codimension.

Using valuation theory we shall give explicit families of $2^{2^{\aleph_0}}$ distinct isomorphism types of subfields of codimension 2 in $\mathbb{C}$ and answer corresponding questions for arbitrary algebraically closed fields of characteristic 0.

Recall that a field F with a valuation v is called maximally complete if the prolongation of v to any proper extension field of F has either strictly larger residue class field or strictly larger value group. V is called a rank one valuation if its value group is a subgroup of the additive group of the real numbers.

Lemma. *There exist $2^{2^{\aleph_0}}$ non-isomorphic ordered divisible subgroups of the additive (ordered) group of the real numbers $\mathbb{R}$.*

PROOF. Let $\{t_\alpha\}$ be a transcendence basis of $\mathbb{R}$ with respect to $\mathbb{Q}$. For any subset I of $\{t_\alpha\}$ let G_I be the divisible subgroup of $\mathbb{R}$ generated by 1 and $\{t_\alpha\}$, $t_\alpha \in I$. Since isomorphisms of ordered divisible subgroups of $\mathbb{R}$ are homotheties, the G_I 's are readily checked to be non-isomorphic ordered groups. Since $\text{Card}\{t_\alpha\} = 2^{\aleph_0}$ the family $\{G_I\}$ has cardinality $2^{2^{\aleph_0}}$.

Theorem 1. *To any maximally complete rank one valuation w of the complex number field $\mathbb{C}$ for which the residue class field has characteristic 0 there corresponds a subfield K_w of codimension 2 in $\mathbb{C}$ such that the restriction of w to K_w is maximally complete. Here $K_{w_1} \neq K_{w_2}$ if w_1 and w_2 are inequivalent. If K_{w_1} and K_{w_2} are (algebraically) isomorphic, w_1 is equivalent to $w_2 \circ \varphi$ for a suitable automorphism φ of the complex field $\mathbb{C}$.*

PROOF. Let $\mathbb{C}$ be maximally complete with respect to w and let G be the value group. The residue class field L is an algebraically closed field of charac-

teristic 0. By the structure theorem for maximally complete fields (cf. SCHILLING [6]) we obtain that with respect to a suitable factor set from G to L the field C is analytically isomorphic to a formal power series field over L with exponents in G . Since the multiplicative group of L is divisible we may assume the factor set is identically 1. Hence C is analytically isomorphic to the formal power series field $L((G))$ consisting of all power series $l(x) = \sum_g l_g x^g$, $l_g \in L$, $g \in G$, where the support $\text{supp}(l(x)) = \{g \in G \mid l_g \neq 0\}$ is a well-ordered subset of G . Since L is algebraically closed of characteristic 0 and $\text{Card}(L) \leq 2^{\aleph_0}$, L may be taken to be an algebraically closed subfield of C . $L \cap R$ has then codimension 2 in L and $(L \cap R)((G))$ is a subfield of codimension 2 in C which is maximally complete with respect to w .

It is well known that a field is algebraically closed if it is complete with respect to two non-equivalent rank one valuations. This gives the second assertion of the theorem. Finally, if σ is an (algebraical) isomorphism from K_{w_1} to K_{w_2} K_{w_1} is complete with respect to the valuations w_1 and $w_2 \sigma$ which are thus equivalent on K_{w_1} . σ extends to an automorphism φ of $C \cdot w_1$ and $w_2 \varphi$ are thus prolongations of valuations which are equivalent on K_{w_1} . Since K_{w_1} in particular is henselian, w_1 and $w_2 \varphi$ are equivalent on C .

Corollary. There exist exactly $2^{2^{\aleph_0}}$ non-isomorphic (non-archimedean ordered) subfields of codimension 2 in C .

PROOF. For any ordered divisible subgroup G of R there exists a valuation of C with value group G for which C is maximally complete and the corresponding residue class field has characteristic 0. By the lemma there are $2^{2^{\aleph_0}}$ such groups G and the corresponding subfields of C are non-isomorphic by theorem 1 (and readily checked to be non-archimedean ordered). This proves the corollary since obviously there cannot be more than $2^{2^{\aleph_0}}$ subfields of C .

Remark 1. It is easy to see that there are $2^{2^{\aleph_0}}$ non-isomorphic archimedean ordered subfields of codimension 2 in C . The intersection of all such subfields is the field of all totally real algebraic numbers.

Remark 2. The above proofs show that if G runs through $2^{2^{\aleph_0}}$ non-isomorphic subgroups of R the power series fields $R((G))$ form $2^{2^{\aleph_0}}$ distinct isomorphism types of fields embeddable with codimension 2 in C .

Remark 3. An explicit family of non-isomorphic fields of codimension 2 in C can be obtained as follows: If $L_0 = R$, $L_{i+1} = \bigcup_{n=1}^{\infty} L_i((x^{\frac{1}{n}}))$, $i \geq 0$, then the fields L_i are non-isomorphic and embeddable with codimension 2 in C . Each L_i is henselian but not complete with respect to any valuation.

Next we consider arbitrary algebraically closed fields of characteristic 0.

Theorem 2. Let K be an algebraically closed field of characteristic 0. If K is an algebraic extension of the rational number field Q there is up to isomorphism just one subfield of codimension 2 in K . Otherwise there are exactly $2^{\text{Card}(K)}$ non-isomorphic subfields of codimension 2 in K . Moreover, for any subfield L of codimension 2 in K there are $2^{\text{Card}(K)}$ isomorphic subfields of codimension 2 in K .

PROOF. Let t be the transcendency degree of K with respect to $\mathbf{Q}$. If $t=0$ any real closed subfield of K is isomorphic to the field of all real algebraic numbers, and the restriction to any such subfield of the $2^{\aleph_0}$ automorphisms of K give rise to $2^{\aleph_0}$ isomorphic copies.

If $t>0$ we first consider the case where t is finite. Let $\{\alpha_i\}$, $(\text{Card}(\{\alpha_i\})=2^{\aleph_0})$ be a transcendency basis for the real number field $\mathbf{R}$ over $\mathbf{Q}$. $\{\alpha_i\}$ contains $2^{\aleph_0}$ mutually disjoint subsets A_j each consisting of t elements. Let K_j be the real closure of $\mathbf{Q}(\alpha_v)$, $(\alpha_v \in A_j)$ in $\mathbf{R}$ with respect to the order induced by $\mathbf{R}$. The ordered fields K_j are obviously pairwise non-isomorphic. The algebraic closure of each K_j is an extension of degree 2 and is isomorphic to K ; hence the first statement of the theorem is proved when t is finite. As for the second statement let L be a subfield of condimension 2 in K . Let $\beta_1, \dots, \beta_t$ be a transcendency basis for $L/\mathbf{Q}$. There is an automorphism σ of $\mathbf{Q}(\beta_1, \dots, \beta_t)$ which does not preserve the order that $L(\cong \mathbf{Q}(\beta_1, \dots, \beta_t))$ has. σ extends to $2^{\aleph_0}$ distinct automorphisms of $K/\mathbf{Q}$ whose restrictions to L give rise to $2^{\aleph_0}$ isomorphic copies of L in K .

Next consider the case where t is an infinite cardinal number $\aleph$.

Clearly $\text{Card}(K)=\aleph$. Let G be the direct sum

$$G = \sum_{\alpha < \aleph} \oplus A_\alpha e_\alpha, \quad A_\alpha = \mathbf{Z} \quad \text{or} \quad \mathbf{Z} + \mathbf{Z}\sqrt{2}$$

where α runs through all ordinal numbers $< \aleph$. We order G lexicographically. Let $F = \mathbf{Q}((G))$ be the field of all formal power series on G over $\mathbf{Q}$. Any element $k \in F$ can be written $k = \sum_g k_g x^g$, $k_g \in \mathbf{Q}$ where the support of k $\text{supp}(k) = \{g \in G \mid k_g \neq 0\}$ is a wellordered subset of G . F can be ordered in the obvious lexicographic way.

Generally, to any (non-archimedean) ordering of a field corresponds a valuation whose valuation ring consists of all elements which are not infinitely greater than 1 (cf. [2] Ex. 3 p. 171).

For $F = \mathbf{Q}((G))$ this valuation w_G is defined by $w_G(k) = \min \{g\}$, $(g \in \text{supp}(k))$. Let $B = \{\alpha \mid A_\alpha = \mathbf{Z}\}$, $C = \{\alpha \mid A_\alpha = \mathbf{Z} + \mathbf{Z}\sqrt{2}\}$ and $D = \bigcup_{\alpha \in B} e_\alpha \cup \bigcup_{\alpha \in C} e_\alpha \cup \bigcup_{\alpha \in C} \sqrt{2}e_\alpha$. Obviously $\text{Card}(D)=\aleph$. Let $\{u_d\}$, $d \in D$ be a transcendency basis for K over $\mathbf{Q}$ and set $M = \mathbf{Q}(\{u_d\})$. Define a mapping φ from M to F by $\varphi(u_d) = X^d$ and $\varphi(q) = q$, $q \in \mathbf{Q}$. Hereby one gets an isomorphism of M onto a subfield of F and thus an ordering of M . The corresponding valuation is $w_G \varphi$. G is the value group of w_G . Let M_G be a real closure in K of the ordered field M . M_G is an ordered field whose corresponding valuation has the value group $G \otimes \mathbf{Q} = \sum_{\alpha < \aleph} \oplus (A_\alpha \otimes \mathbf{Q}) l_\alpha$ where $A_\alpha \otimes \mathbf{Q} = \mathbf{Q}$ or $\mathbf{Q} + \mathbf{Q}\sqrt{2}$ according as $\alpha \in B$ or $\alpha \in C$.

In the definition of G we have for each α two choices for A_α and we thus get $2^\aleph$ ordered groups. By considering skeletons (cf. Fuchs [3] or Ribenboim [5]) it is easily seen that the corresponding groups $G \otimes \mathbf{Q}$ are pairwise non-isomorphic (qua ordered groups). This implies that the corresponding real closed fields M_G are non-isomorphic since any isomorphism would be order-preserving; the corresponding valuations w_G would then be equivalent and in particular have order-isomorphic value groups.

Finally, let L be a subfield of codimension 2 in K . If $\{\beta_i\}$ is a transcendence basis for L over $\mathbf{Q}$ there are $2^{\aleph}$ permutations of $\{\beta_i\}$ which extend to $2^{\aleph}$ automorphisms of $K/\mathbf{Q}$. The restrictions of these automorphisms to L give rise to $2^{\aleph}$ distinct isomorphic copies of L having codimension 2 in K . Theorem 2 is now proved.

Remark. It is not hard to show that the intersection of all subfields of codimension 2 in K is the field of all totally real algebraic numbers (i.e. algebraic numbers all of whose conjugates are real). One could also mention, that a subfield of codimension 2 in K has a trivial automorphism group if and only if it is archimedean ordered. In fact, any nonarchimedean ordered real closed field L has at least $\text{Card}(L)$ distinct automorphisms.

BIALYNICKI-BIRULA [1] proved that any algebraically closed field has a subfield of countable condimension. We shall finish by proving the following

Theorem 3. *Let K be an arbitrary algebraically closed field of characteristic 0. If $\text{Card}(K) \cong 2^{\aleph_0}$ and $\aleph$ an infinite cardinal number $< \text{Card}(K)$ there exists a subfield L of K for which $[K:L] = \aleph$.*

Corollary. *Assuming the continuum hypothesis an algebraical closed field of characteristic 0 has subfield of any prdscribed infinite codimension $< \text{Card}(K)$.*

PROOF. (of Theorem 3). Let $\text{Card}(K) = \overline{\aleph}$ and let G be the lexicographically ordered vector space over the rational field $\mathbf{Q}$:

$$G = \sum_{\alpha < \overline{\aleph}} \mathbf{Q}e_\alpha$$

where α runs through all ordinals $< \overline{\aleph}$. For any finite set I of ordinals $< \overline{\aleph}$ let F_I be the subfield of the formal power series field $\mathbf{C}((G))$ consisting of all power series of the form $\sum c_g x^g$ $c_g \in \mathbf{C}$ where the exponents g belong to the $\mathbf{Q}$ -space generated by $\{e_\alpha\}$, $\alpha \in I$ and the denominators of q_α in the expression $g = \sum g_\alpha e_\alpha$, $q_\alpha \in \mathbf{Q}$, are bounded.

$\mathbf{C}((G))$ is a maximally complete field with algebraically closed residue class field and divisible value group and hence algebraically closed. F_I is algebraically closed in $\mathbf{C}((G))$ and consequently itself an algebraically closed field.

For an arbitrary set I of ordinals $< \overline{\aleph}$ we define $F_I = \bigcup_{I'} F_{I'}$ where I' runs through all finite subsets of I . Further we define F_I^* as the subfield of F_I of all power series whose exponents g have "integral coefficients", i.e. belong to $\sum_{\alpha < \overline{\aleph}} \mathbf{Z}e_\alpha$.

Now let J be the set of all ordinals $< \overline{\aleph}$. Then $\text{Card}(F_J) = \text{Card}(J) = \overline{\aleph}$ since $\overline{\aleph} > 2^{\aleph_0}$. Hence $F_J \simeq K$. We now choose a subset I of J such that $\text{Card}(J \setminus I) = \aleph$ and set $L = F_I^* \cap F_I$. The elements x^g , $g = \sum q_\alpha e_\alpha$ (finite sum), $\alpha \in J \setminus I$ where the q_α 's run through a set of representatives of the rational numbers modulo the integers, form a basis for F_J over L . Thus $[F_J:L] = \aleph$ and the theorem is proved in view of the isomorphism $F_J \simeq K$.

References

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