Non-additive ring and module theory II.
$\mathscr{C}$ - Categories, $\mathscr{C}$ - Functors and $\mathscr{C}$ - Morphisms

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Consider the example (Cat, $\times$, $\mathbf{1}$) of a monoidal category. A monoid in this category is a (small) category $\mathscr{C}$ together with functors $\otimes:\mathscr{C}\times\mathscr{C}\to\mathscr{C}$ and $\mathscr{T}:\mathbf{1}\to\mathscr{C}$ (where we use the notation $\mathscr{T}(e)=I$) such that the diagrams

$$(*)\qquad \begin{tikzcd} \mathscr{C}\times\mathscr{C}\times\mathscr{C} \arrow[r, "\mathscr{C}\times\otimes"] \arrow[d, "\otimes\times\mathscr{C}"'] & \mathscr{C}\times\mathscr{C} \arrow[d, "\otimes"] \\ \mathscr{C}\times\mathscr{C} \arrow[r, "\otimes"] & \mathscr{C} \end{tikzcd} \qquad . \qquad \begin{tikzcd} \mathbf{1}\times\mathscr{C}\cong\mathscr{C}\cong\mathscr{C}\times\mathbf{1} \arrow[r, "\mathscr{C}\times\mathscr{T}"] \arrow[d, "\mathscr{T}\times\mathscr{C}"] \arrow[dr, "id_{\mathscr{C}}"] & \mathscr{C}\times\mathscr{C} \arrow[d, "\otimes"] \\ \mathscr{C}\times\mathscr{C} \arrow[r, "\otimes"] & \mathscr{C} \end{tikzcd}$$

commute. This means $A\otimes(B\otimes C)=(A\otimes B)\otimes C$ for all $A, B, C\in\mathscr{C}$ and $A\otimes I= =A=I\otimes A$ for all $A\in\mathscr{C}$. These identities are natural transformations in A, B, C. Such a category $\mathscr{C}$ is called a *strictly monoidal category*. For the general case the two diagrams (*) are commutative up to a natural isomorphism and such that the coherence conditions of § 1. hold. By the coherence theorem of [5] this implies that all morphisms composed of $\alpha:A\otimes(B\otimes C)\cong(A\otimes B)\otimes C$, $\lambda:I\otimes A\cong A$, $\varrho:A\otimes I\cong A$, identities and $\otimes$ which formally have common domain and codomain are equal.

Now consider an object $\mathscr{M}$ in (Cat, $\times$, $\mathbf{1}$) on which a strict monoidal category $\mathscr{C}$ operates in the right. Then $\mathscr{M}$ is a category together with a functor $\otimes:\mathscr{M}\times\mathscr{C}\to\mathscr{M}$, such that

$$M\otimes(C\otimes D)=(M\otimes C)\otimes D \quad \text{for all} \quad M\in\mathscr{M}, \quad C, D\in\mathscr{C}$$

and

$$M\otimes I = M \quad \text{for all} \quad M\in\mathscr{M}.$$

These identities are natural transformations in M, C, D. Such a category $\mathscr{M}$ will be called a *strict $\mathscr{C}$-category*. A useful generalization of this is a *$\mathscr{C}$-category $\mathscr{M}$* for an arbitrary monoidal category $\mathscr{C}$. This is a category $\mathscr{M}$ together with a functor $\otimes:\mathscr{M}\times\mathscr{C}\to\mathscr{M}$ and natural isomorphisms $\beta:M\otimes(C\otimes D)\cong(M\otimes C)\otimes D$ and $\sigma:M\otimes I\cong M$ such that all formal diagrams composed of $\alpha, \beta, \sigma, \lambda, \varrho$, their inverses, $\otimes$ in $\mathscr{C}$, and $\otimes$ with respect to $\mathscr{M}$, identities, and compositions commute.

In particular we require the commutativity of the following diagrams:

$$\begin{tikzcd}
M\otimes(C\otimes(D\otimes E)) \arrow[rr, "\beta"] \arrow[d, "M\otimes\alpha"] & & (M\otimes C)\otimes(D\otimes E) \arrow[d, "\beta"] \\
M\otimes((C\otimes D)\otimes E) \arrow[r, "\beta"] & (M\otimes(C\otimes D))\otimes E \arrow[r, "\beta\otimes E"] & ((M\otimes C)\otimes D)\otimes E
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes(I\otimes C) \arrow[rr, "\beta"] \arrow[dr, "M\otimes\lambda"'] & & (M\otimes I)\otimes C \arrow[dl, "\tilde{\sigma}\otimes C"] \\
& M\otimes C &
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes(C\otimes I) \arrow[rr, "\beta"] \arrow[dr, "M\otimes\varrho"'] & & (M\otimes C)\otimes I \arrow[dl, "\tilde{\sigma}"] \\
& M\otimes C &
\end{tikzcd}$$

If $\mathscr{C}$ is a symmetric monoidal category (which corresponds to the notion of a commutative monoid in Cat), then we also require the commutativity of

$$\begin{tikzcd}
M\otimes(I\otimes I) \arrow[r, "M\otimes\tau"] \arrow[dr, "\beta"'] & M\otimes(I\otimes I) \arrow[d, "\beta"] \\
& (M\otimes I)\otimes I
\end{tikzcd}$$

Now let us regard the corresponding morphisms. A morphism of monoids in Cat is a functor $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ of strictly monoidal categories $\mathscr{C}$ and $\mathscr{D}$, such that

$$\mathscr{F}(C\otimes D)=\mathscr{F}(C)\otimes\mathscr{F}(D)$$

and

$$\mathscr{F}(I)=J$$

where $I \in \mathscr{C}$ and $J \in \mathscr{D}$ are the neutral objects. The identities are natural transformations in C and D. If $\mathscr{C}$ and $\mathscr{D}$ are just monoidal categories then we require natural isomorphisms

$$\delta : \mathscr{F}(C \otimes D) \cong \mathscr{F}(C) \otimes \mathscr{F}(D)$$

$$\zeta : \mathscr{F}(I) \cong J$$

such that the following diagrams commute:

$$\begin{tikzcd}
\mathscr{F}(C \otimes I) \arrow[r, "\delta"] \arrow[rrd, "\mathscr{F}(\varrho)"'] & \mathscr{F}(C) \otimes \mathscr{F}(I) \arrow[r, "\mathscr{F}(C) \otimes \zeta"] & \mathscr{F}(C) \otimes J \arrow[d, "\varrho"] \\
& & \mathscr{F}(C)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{F}(I \otimes C) \arrow[r, "\delta"] \arrow[rrd, "\mathscr{F}(\lambda)"'] & \mathscr{F}(I) \otimes \mathscr{F}(C) \arrow[r, "\zeta \otimes \mathscr{F}(C)"] & J \otimes \mathscr{F}(C) \arrow[d, "\lambda"] \\
& & \mathscr{F}(C)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{F}(C \otimes (D \otimes E)) \arrow[r, "\delta"] \arrow[d, "\mathscr{F}(\alpha)"] & \mathscr{F}(C) \otimes \mathscr{F}(D \otimes E) \arrow[r, "\mathscr{F}(C) \otimes \delta"] & \mathscr{F}(C) \otimes (\mathscr{F}(D) \otimes \mathscr{F}(E)) \arrow[d, "\alpha"] \\
\mathscr{F}((C \otimes D) \otimes E) \arrow[r, "\delta"] & \mathscr{F}(C \otimes D) \otimes \mathscr{F}(E) \arrow[r, "\delta \otimes \mathscr{F}(E)"] & (\mathscr{F}(C) \otimes \mathscr{F}(D)) \otimes \mathscr{F}(E).
\end{tikzcd}$$

Such a functor is called a *monoidal functor*. If $\mathscr{C}$ and $\mathscr{D}$ are symmetric we require in addition the commutativity of

$$\begin{tikzcd}
\mathscr{F}(C \otimes D) \arrow[r, "\mathscr{F}(\gamma)"] \arrow[d, "\delta"] & \mathscr{F}(D \otimes C) \arrow[d, "\delta"] \\
\mathscr{F}(C) \otimes \mathscr{F}(D) \arrow[r, "\gamma"] & \mathscr{F}(D) \otimes \mathscr{F}(C).
\end{tikzcd}$$

Let $\mathscr{M}$ and $\mathscr{N}$ be right $\mathscr{C}$-objects in Cat for a strictly monoidal category $\mathscr{C}$. A $\mathscr{C}$-morphism $\mathscr{F} : \mathscr{M} \to \mathscr{N}$ is a functor such that

$$\mathscr{F}(M \otimes C) = \mathscr{F}(M) \otimes C,$$

where the identity is a natural transformation in M and C. In the general case of $\mathscr{C}$-categories $\mathscr{M}$ and $\mathscr{N}$ for a monoidal category $\mathscr{C}$, a *$\mathscr{C}$-functor* is a functor $\mathscr{F} : \mathscr{M} \to \mathscr{N}$ together with a natural isomorphism

$$\xi : \mathscr{F}(M \otimes C) \cong \mathscr{F}(M) \otimes C$$

such that the following diagrams commute:

$$\begin{tikzcd}
\mathscr{F}(M \otimes (C \otimes D)) \arrow[rr, "\xi"] \arrow[d, "\mathscr{F}(\beta)"] & & \mathscr{F}(M) \otimes (C \otimes D) \arrow[d, "\beta"] \\
\mathscr{F}((M \otimes C) \otimes D) \arrow[r, "\xi"] & \mathscr{F}(M \otimes C) \otimes D \arrow[r, "\xi \otimes D"] & (\mathscr{F}(M) \otimes C) \otimes D
\end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}(M \otimes I) \arrow[rr, "\xi"] \arrow[dr, "\mathcal{F}(\sigma)"'] & & \mathcal{F}(M) \otimes I \arrow[dl, "\sigma"] \\ & \mathcal{F}(M) & \end{tikzcd}$$

Finally we introduce natural transformations between monoidal functors, resp. between $\mathscr{C}$-functors. Let $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ and $\mathscr{G}:\mathscr{C}\to\mathscr{D}$ be monoidal functors. A natural transformation $\chi:\mathscr{F}\to\mathscr{G}$ is called a *monoidal transformation* if

$$\begin{array}{ccc} \mathscr{F}(C\otimes D) & \xrightarrow{\delta} & \mathscr{F}(C)\otimes\mathscr{F}(D) \\ \downarrow \chi & & \downarrow \chi\otimes\chi \\ \mathscr{G}(C\otimes D) & \xrightarrow{\delta} & \mathscr{G}(C)\otimes\mathscr{G}(D) \end{array}$$

and

$$\begin{tikzcd} \mathcal{F}(I) \arrow[d, "\chi"'] \arrow[dr, "\zeta"] & \\ \mathcal{G}(I) \arrow[r, "\zeta"'] & J \end{tikzcd}$$

commute.

A natural transformation $\psi:\mathscr{F}\to\mathscr{G}$ between $\mathscr{C}$-functors $\mathscr{F}:\mathscr{M}\to\mathscr{N}$ and $\mathscr{G}:\mathscr{M}\to\mathscr{N}$ is called a *$\mathscr{C}$-morphism* if

$$\begin{array}{ccc} \mathscr{F}(M\otimes C) & \xrightarrow{\xi} & \mathscr{F}(M)\otimes C \\ \downarrow \psi & & \downarrow \psi\otimes C \\ \mathscr{G}(M\otimes C) & \xrightarrow{\xi} & \mathscr{G}(M)\otimes C \end{array}$$

commutes.

Now we can introduce the notion of a right A-object in a right $\mathscr{C}$-category $\mathscr{D}$, where A is a monoid in C. An object $M\in\mathscr{D}$ together with $\nu_M: M\otimes A\to M$ is an *A-object* if the diagrams

$$\begin{array}{ccccc} M\otimes(A\otimes A) & \overset{\beta}{\cong} & (M\otimes A)\otimes A & \xrightarrow{\nu_M\otimes A} & M\otimes A \\ \downarrow M\otimes\mu_A & & & & \downarrow \nu_M \\ M\otimes A & & \xrightarrow{\qquad\nu_M\qquad} & & M \end{array}$$

and

$$\begin{tikzcd} M \overset{\rho}{\cong} M\otimes I \arrow[r, "M\otimes\eta"] \arrow[dr, "\mathrm{id}_M"'] & M\otimes A \arrow[d, "\nu_M"] \\ & M \end{tikzcd}$$

commute. It is clear how *A-morphisms* are to be defined. Thus we get a category of A-objects $\mathscr{D}_A$.

4.1. **Lemma** *Let $\mathscr{F}:\mathscr{D}\to\mathscr{D}'$ be a $\mathscr{C}$-functor. Then $\mathscr{F}$ induces a functor $\mathscr{F}_A:\mathscr{D}_A\to\mathscr{D}'_A$.*

Proof. Let $(M, v_M)\in\mathscr{D}_A$. We define $\mathscr{F}_A(M, v_M):=(\mathscr{F}(M), \mathscr{F}'(v_M))$ where $\mathscr{F}'(v_M)$ is the morphism $\mathscr{F}(M)\otimes A\cong\mathscr{F}(M\otimes A)\xrightarrow{\mathscr{F}(v_M)}\mathscr{F}(M)$. The diagrams

$$\begin{tikzcd}
\mathscr{F}(M)\otimes(A\otimes A) \arrow[r, phantom, "\cong"] \arrow[dd, phantom, "\cong"] & (\mathscr{F}(M)\otimes A)\otimes A \arrow[r, "\mathscr{F}'(v_M)\otimes A"] \arrow[d, phantom, "\cong"] & \mathscr{F}(M)\otimes A \arrow[dd, phantom, "\cong"] \\
& \mathscr{F}(M\otimes A)\otimes A \arrow[ur, "\mathscr{F}(v_M)\otimes A"'] \arrow[d, phantom, "\cong"] & \\
\mathscr{F}(M\otimes(A\otimes A)) \arrow[r, phantom, "\cong"] \arrow[d, "\mathscr{F}(M\otimes\mu_A)"] & \mathscr{F}((M\otimes A)\otimes A) \arrow[r, "\mathscr{F}(v_M\otimes A)"'] & \mathscr{F}(M\otimes A) \arrow[d, "\mathscr{F}(v_M)"] \\
\mathscr{F}(M\otimes A) \arrow[rr, "\mathscr{F}(v_M)"'] & & \mathscr{F}(M)
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathscr{F}(M)\otimes I \arrow[r, "\mathscr{F}(M)\otimes\eta"] \arrow[d, phantom, "\cong"] & \mathscr{F}(M)\otimes A \arrow[d, phantom, "\cong"] \\
\mathscr{F}(M) \arrow[ur, phantom, "\cong"] \arrow[r, phantom, "\cong"] \arrow[drr, "id_{\mathscr{F}(M)}"'] & \mathscr{F}(M\otimes I) \arrow[r, "\mathscr{F}(M\otimes\eta)"] & \mathscr{F}(M\otimes A) \arrow[d, "\mathscr{F}(v_M)"] \\
& & \mathscr{F}(M)
\end{tikzcd}$$

commute. Similarly if $f:M\to N$ is a morphism in $\mathscr{D}_A$ then $\mathscr{F}(f)$ is in $\mathscr{D}'_A$ since

$$\begin{array}{ccc}
\mathscr{F}(M)\otimes A & \xrightarrow{\mathscr{F}(f)\otimes A} & \mathscr{F}(N)\otimes A \\
\wr\wr & & \wr\wr \\
\mathscr{F}(M\otimes A) & \xrightarrow{\mathscr{F}(f\otimes A)} & \mathscr{F}(N\otimes A) \\
\mathscr{F}(v_M)\downarrow & & \downarrow\mathscr{F}(v_N) \\
\mathscr{F}(M) & \xrightarrow{\mathscr{F}(f)} & \mathscr{F}(N)
\end{array}$$

commutes. So define $\mathscr{F}_A(f):=\mathscr{F}(f)$ in $\mathscr{D}'_A$ and we get a functor $\mathscr{F}_A:\mathscr{D}_A\to\mathscr{D}'_A$.

This proof shows already why we had to require certain coherence conditions for the definition of $\mathscr{C}$-functors. The most important property of $\mathscr{C}$-functors which is very similar to properties of exact functors will be discussed later on. First we need a few additional properties of A-objects in $\mathscr{C}$.

By Theorem 2.2 and the remark following 2.3 Lemma 3 of [12] the following is a difference cokernel of a contractible pair in $\mathscr{C}$

$$A\otimes(A\otimes M)\underset{A\otimes v_M}{\overset{(\mu_A\otimes M)\circ\alpha}{\rightrightarrows}}A\otimes M\xrightarrow{v_M}M$$

for each A-object M in $\mathscr{C}$. In fact only the contraction morphism $(\eta\otimes(A\otimes M))\cdot\lambda^{-1}:A\otimes M\to A\otimes(A\otimes M)$ is in $\mathscr{C}$, the morphisms of the above sequence are even in ${}_A\mathscr{C}$. So we have a difference cokernel of a $(\mathscr{U}:{}_A\mathscr{C}\to\mathscr{C})$-contractible pair in ${}_A\mathscr{C}$.

4.2. **Theorem.** *Let* $\mathscr{F}: {}_A\mathscr{C} \to {}_B\mathscr{C}$ *be a covariant functor. Equivalent are*

a) $\mathscr{F}$ *is a* $\mathscr{C}$*-functor and* $\mathscr{F}$ *preserves difference cokernels of* $\mathscr{U}$*-contractible pairs.*

b) *There is a* $B-A$*-biobject* Q *which is* A*-coflat (which is unique up to an isomorphism) such that* $\mathscr{F} \cong Q \otimes_A$ *as functors from* ${}_A\mathscr{C}$ *to* ${}_B\mathscr{C}$.

PROOF. Assume that a) holds. Then we have a difference cokernel

$$\mathscr{F}(A \otimes (A \otimes M)) \rightrightarrows \mathscr{F}(A \otimes M) \to \mathscr{F}(M).$$

Since $\mathscr{F}$ is a $\mathscr{C}$-functor we get a difference cokernel

$$\mathscr{F}(A) \otimes (A \otimes M) \underset{\mathscr{F}(A) \otimes \nu_M}{\overset{(\mathscr{F}'(\mu_A) \otimes M) \circ \alpha}{\rightrightarrows}} \mathscr{F}(A) \otimes M \xrightarrow{\mathscr{F}(\nu_M) \circ \xi^{-1}} \mathscr{F}(M).$$

By definition of $\mathscr{F}(A) \otimes_A M$ in §3 we get a natural isomorphism $\mathscr{F}(M) \cong \mathscr{F}(A) \otimes_A M$ with the $B-A$-biobject $\mathscr{F}(A) = Q$. If $Q' \otimes_A \cong \mathscr{F}$ then $Q' \otimes_A A \cong Q \otimes_A A$ hence $Q' \cong Q$ as $B-A$-biobjects.

Conversely assume that b) holds. The following commutative diagram indicates that $Q \otimes_A : {}_A\mathscr{C} \to {}_B\mathscr{C}$ is a $\mathscr{C}$-functor:

$$\begin{array}{ccccc}
(Q \otimes (A \otimes M)) \otimes X & \rightrightarrows & (Q \otimes M) \otimes X & \to & (Q \otimes_A M) \otimes X \\
\wr\| & & \wr\| & & \wr\| \\
Q \otimes (A \otimes (M \otimes X)) & \rightrightarrows & Q \otimes (M \otimes X) & \to & Q \otimes_A (M \otimes X).
\end{array}$$

The verification of the coherence diagrams for $\mathscr{C}$-functors is left to the reader. Now let

$$M \underset{f_1}{\overset{f_0}{\rightrightarrows}} N \xrightarrow{h} P$$

be a difference cokernel of a $\mathscr{U}$-contractible pair in ${}_A\mathscr{C}$ with contraction $g: N \to M$ in $\mathscr{C}$ and section $k: P \to N$ in $\mathscr{C}$ such that $f_0 g = id_N$, $f_1 g f_0 = f_1 g f_1$, $hk = id_P$ and $kh = f_1 g$. Now $Q \otimes$ preserves the whole situation, so we get a commutative diagram

$$\begin{tikzcd}
Q \otimes N \arrow[rr, "id_{Q \otimes N}"] \arrow[dr, "Q \otimes g"] \arrow[ddd, "Q \otimes h"'] & & Q \otimes N \arrow[ddd, "Q \otimes h"] \\
& Q \otimes M \arrow[ur, "Q \otimes f_0"'] \arrow[d, "Q \otimes f_1"] & \\
& Q \otimes N \arrow[dr, "Q \otimes h"] & \\
Q \otimes P \arrow[ur, "Q \otimes k"] \arrow[rr, "id_{Q \otimes P}"'] & & Q \otimes P
\end{tikzcd}$$

This implies that

$$Q \otimes M \underset{Q \otimes f_1}{\overset{Q \otimes f_0}{\rightrightarrows}} Q \otimes N \xrightarrow{Q \otimes h} Q \otimes P$$

is a difference kokernel of a $\mathscr{U}$-contractible pair [12, 2.3].
A similar diagram is obtained by tensoring with $Q \otimes A$ on the left. This induces a commutative diagram.

$$\begin{array}{ccccc} (Q \otimes A) \otimes M & \rightrightarrows & (Q \otimes A) \otimes N & \to & (Q \otimes A) \otimes P \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ Q \otimes M & \rightrightarrows & Q \otimes N & \to & Q \otimes P \\ \downarrow & & \downarrow & & \downarrow \\ Q \otimes_A M & \rightrightarrows & Q \otimes_A N & \to & Q \otimes_A P \end{array}$$

where all rows and columns are difference cokernels in ${}_B\mathscr{C}$, due to the fact that f_0, f_1 and h are A-morphisms and that colimits commute with colimits. Hence $Q \otimes_A : {}_A\mathscr{C} \to {}_B\mathscr{C}$ preserves difference cokernels of $\mathscr{U}$-contractible pairs.

It seems that the strong property of being a $\mathscr{C}$-functor rarely occurs. But the following theorem shows that this property is closely related to inner hom functors. Let us first consider the case $\mathscr{C} = \mathbf{Z}-\mathrm{Mod}$. Let $\mathscr{F} : {}_A\mathscr{C} \to {}_B\mathscr{C}$ have a right adjoint $\mathscr{G} : {}_B\mathscr{C} \to {}_A\mathscr{C}$. Then $\mathscr{F}$ and $\mathscr{G}$ preserve colimits resp. limits hence they are additive functors. Now the natural bijection

$${}_B\mathscr{C}(\mathscr{F}(M), N) \cong {}_A\mathscr{C}(M, \mathscr{G}(N)) \quad \text{is composed of}$$

$$\mathscr{G} : {}_B\mathscr{C}(\mathscr{F}(M), N) \to {}_A\mathscr{C}(\mathscr{G}\mathscr{F}(M), \mathscr{G}(N))$$

and

$${}_A\mathscr{C}(\Phi M, \mathscr{G}(N)) : {}_A\mathscr{C}(\mathscr{G}\mathscr{F}(M), \mathscr{G}(N)) \to {}_A\mathscr{C}(M, \mathscr{G}(N)).$$

Both maps are homomorphisms of abelian groups since $\mathscr{G}$ is an additive functor and $\Phi M : M \to \mathscr{G}\mathscr{F}(M)$ is in ${}_A\mathscr{C}$. Thus a pair of adjoint functors between ${}_A\mathscr{C}$ and ${}_B\mathscr{C}$ with $\mathscr{C} = \mathbf{Z}-\mathrm{Mod}$ is automatically such that not only the morphism sets ${}_B\mathscr{C}(\mathscr{F}(M), N)$ and ${}_A\mathscr{C}(M, \mathscr{G}(N))$ are isomorphic in the category of sets but even the inner hom functors ${}_B[\mathscr{F}(M), N]$ and ${}_A[M, \mathscr{G}(N)]$ are isomorphic in $\mathscr{C} = \mathbf{Z}-\mathrm{Mod}$. Unfortunately this is not true in the general case. However the following theorem holds

4.3. **Theorem.** a) *Let $\mathscr{C}$ be a closed monoidal category and $\mathscr{F} : {}_A\mathscr{C} \to {}_B\mathscr{C}$ and $\mathscr{G} : {}_B\mathscr{C} \to {}_A\mathscr{C}$ be functors. $\mathscr{F}$ is left adjoint to $\mathscr{G}$ and a $\mathscr{C}$-functor if and only if there is a natural isomorphism*

$${}_B[\mathscr{F}(M), N] \cong {}_A[M, \mathscr{G}(N)]$$

for all $M \in {}_A\mathscr{C}$ and $N \in {}_B\mathscr{C}$.

b) *Let $\mathscr{C}$ be a coclosed monoidal category and $\mathscr{F} : {}_A\mathscr{C} \to {}_B\mathscr{C}$ and $\mathscr{G} : {}_B\mathscr{C} \to {}_A\mathscr{C}$ be functors. $\mathscr{F}$ is right adjoint to $\mathscr{G}$ and a $\mathscr{C}$-functor if and only if there is a natural isomorphism*

$${}_B\langle \mathscr{F}(M), N \rangle \cong {}_A\langle M, \mathscr{G}(N) \rangle$$

for all $M \in {}_A\mathscr{C}$ and $N \in {}_B\mathscr{C}$.

PROOF. First assume that there is a natural isomorphism

$$\Phi : {}_B[\mathscr{F}(M), N] \cong {}_A[M, \mathscr{G}(N)].$$

Then $\mathscr{F}$ is left adjoint to $\mathscr{G}$ by ${}_B\mathscr{C}(\mathscr{F}(M), N) \cong {}_B\mathscr{C}(\mathscr{F}(M) \otimes I, N) \cong \cong \mathscr{C}(I, {}_B[\mathscr{F}(M), N]) \cong \mathscr{C}(I, {}_A[M, \mathscr{G}(N)]) \cong {}_A\mathscr{C}(M \otimes I, \mathscr{G}(N)) \cong {}_A\mathscr{C}(M, \mathscr{G}(N))$. Call this isomorphism φ. It is clear that φ is natural in M and N.

Now define $\xi : \mathscr{F}(M \otimes C) \cong \mathscr{F}(M) \otimes C$ by ${}_B\mathscr{C}(\xi, N)$ as ${}_B\mathscr{C}(\mathscr{F}(M) \otimes C, N) \cong \cong \mathscr{C}(C, {}_B[\mathscr{F}(M), N]) \cong \mathscr{C}(C, {}_A[M, \mathscr{G}(N)]) \cong {}_A\mathscr{C}(M \otimes C, \mathscr{G}(N)) \cong {}_B\mathscr{C}(\mathscr{F}(M \otimes C), N)$. Again ξ is clearly a natural transformation in M and C.

To check the two properties for a $\mathscr{C}$-functor denote by

$$\psi_A : {}_A\mathscr{C}(M \otimes C, M') \cong \mathscr{C}(C, {}_A[M, N'])$$

$$\psi_B : {}_B\mathscr{C}(N \otimes C, N') \cong \mathscr{C}(C, {}_B[N, N'])$$

the adjointness isomorphisms of 3.10. Then the diagram

$$\begin{tikzcd}
{}_B\mathscr{C}(\mathscr{F}(M), N) \arrow[rr, "\varphi"] \arrow[d, "{}_B\mathscr{C}(\sigma, N)"] \arrow[dr, "{}_B\mathscr{C}(\mathscr{F}(\sigma), N)"] & & {}_A\mathscr{C}(M, \mathscr{G}(N)) \arrow[d, "{}_A\mathscr{C}(\sigma, \mathscr{G}(N))"] \\
{}_B\mathscr{C}(\mathscr{F}(M) \otimes I, N) \arrow[r, "{}_B\mathscr{C}(\xi, N)"] \arrow[d, "\psi_B"] & {}_B\mathscr{C}(\mathscr{F}(M \otimes I), N) \arrow[r, "\varphi"] & {}_A\mathscr{C}(M \otimes I, \mathscr{G}(N)) \arrow[d, "\psi_A"] \\
\mathscr{C}(I, {}_B[\mathscr{F}(M), N]) \arrow[rr, "\mathscr{C}(I, \Phi)"] & & \mathscr{C}(I, {}_A[M, \mathscr{G}(N)])
\end{tikzcd}$$

commutes, the outer hexagon by definition of φ, the lower pentagon by definition of ξ, the upper right hand quadrangle since φ is a natural transformation and the upper left hand triangle since all morphisms of the diagram are isomorphisms. Hence we get a commutative diagram in ${}_B\mathscr{C}$

$$\begin{tikzcd}
\mathscr{F}(M \otimes I) \arrow[rr, "\xi"] \arrow[dr, "\mathscr{F}(\sigma)"'] & & \mathscr{F}(M) \otimes I \arrow[dl, "\sigma"] \\
& \mathscr{F}(M) &
\end{tikzcd}$$

For the second more complicated diagram for $\mathscr{C}$-functors observe first from § 3 that we have natural isomorphisms

$${}_A[M \otimes C, M'] \cong [C, {}_A[M, M']] \quad \text{and} \quad {}_B[N \otimes C, N'] \cong [C, {}_B[N, N']]$$

and that the diagram

$$\begin{tikzcd}
{[C \otimes D, {}_A[M, M']]} \arrow[r, phantom, "\cong"] \arrow[d, phantom, sloped, "\cong"] & {{}_A[M \otimes (C \otimes D), M']} \arrow[r, phantom, "\cong"] & {{}_A[(M \otimes C) \otimes D, M']} \arrow[d, phantom, sloped, "\cong"] \\
{[D, [C, {}_A[M, M']]]} \arrow[rr, phantom, "\cong"] & & {[D, {}_A[M \otimes C, M']]}
\end{tikzcd}$$

and the corresponding diagram with respect to ${}_B\mathscr{C}$ commute. Thus we get a commutative diagram

$$\begin{array}{ccccc}
{}_A[M\otimes(C\otimes D),\mathscr{G}(N)] & \cong & {}_A[(M\otimes C)\otimes D,\mathscr{G}(N)] & \cong & [D,{}_A[M\otimes C,\mathscr{G}(N)]] \\
\wr\Vert & & & & \wr\Vert \\
[C\otimes D,{}_A[M,\mathscr{G}(N)]] & & \longrightarrow & & [D,[C,{}_A[M,\mathscr{G}(N)]]] \\
\wr\Vert & & & & \wr\Vert \\
[C\otimes D,{}_B[\mathscr{F}(M),N]] & & \longrightarrow & & [D,[C,{}_B[\mathscr{F}(M),N]]] \\
\wr\Vert & & & & \wr\Vert \\
{}_B[\mathscr{F}(M)\otimes(C\otimes D),N] & \cong & {}_B[(\mathscr{F}(M)\otimes C)\otimes D,N] & \cong & [D,{}_B[\mathscr{F}(M)\otimes C,N]].
\end{array}$$

Observe now the following diagram

$$\begin{tikzcd}
{}_A[M\otimes(C\otimes D),\mathscr{G}(N)] \arrow[rr] & & {[C\otimes D,{}_A[M,\mathscr{G}(N)]]} \arrow[r] & {[C\otimes D,{}_B[\mathscr{F}(M),N]]} \\
& {}_B[\mathscr{F}(M\otimes(C\otimes D)),N] \arrow[ul] & & {}_B[\mathscr{F}(M)\otimes(C\otimes D),N] \arrow[ll, "{}_B[\xi,N]"'] \arrow[u] \\
& {}_B[\mathscr{F}((M\otimes C)\otimes D,N] \arrow[u, "{}_B[\mathscr{F}(\beta),N]"'] \arrow[dl] & {}_B[\mathscr{F}(M\otimes C)\otimes D,N] \arrow[l, "{}_B[\xi,N]"'] \arrow[d] & {}_B[\mathscr{F}(M)\otimes C)\otimes D,N] \arrow[l, "{}_B[\xi\otimes D,N]"'] \arrow[u, "{}_B[\beta,N]"'] \arrow[d] \\
{}_A[(M\otimes C)\otimes D,\mathscr{G}(N)] \arrow[uuu] \arrow[drr] & & {[D,{}_B[\mathscr{F}(M\otimes C),N]]} \arrow[d] & {[D,{}_B[\mathscr{F}(M)\otimes C,N]]} \arrow[l] \arrow[d] \\
& & {[D,{}_A[M\otimes C,\mathscr{G}(N)]]} \arrow[dr] & {[D,[C,{}_B[\mathscr{F}(M),N]]]} \arrow[d] \\
& & & {[D,[C,{}_A[M,\mathscr{G}(N)]]]}
\end{tikzcd}$$

The outer frame commutes, since the previous diagram commutes.

The left quadrangle is commutative since Φ is a natural transformation, the right hand square since ψ_B is a natural transformation, the three outer pentagons by the definition of ξ. Since all morphisms are isomorphisms, the inner pentagon commutes also. Hence we get

$$\begin{array}{ccc}
\mathscr{F}(M\otimes(C\otimes D)) & \longrightarrow & \mathscr{F}(M)\otimes(C\otimes D) \\
\downarrow\ \mathscr{F}(\beta) & & \downarrow\ \beta \\
\mathscr{F}((M\otimes C)\otimes D) \xleftarrow{\xi} \mathscr{F}(M\otimes C)\otimes D & \xleftarrow{\xi\otimes D} & (\mathscr{F}(M)\otimes C)\otimes D
\end{array}$$

commutative. This proves one direction of part *a)* of the theorem.

Conversely, let $\mathscr{F}$ be a $\mathscr{C}$-functor and left adjoint to $\mathscr{G}$. Then we have isomorphisms

$$\mathscr{C}(C,{}_B[\mathscr{F}(M),N]) \cong {}_B\mathscr{C}(\mathscr{F}(M)\otimes C,N) \cong {}_B\mathscr{C}(\mathscr{F}(M\otimes C),N) \cong$$
$${}_A\mathscr{C}(M\otimes C,\mathscr{G}(N)) \cong \mathscr{C}(C,{}_A[M,\mathscr{G}(N)])$$

natural in $C \in \mathscr{C}$, $M \in {}_A\mathscr{C}$ and $N \in {}_B\mathscr{C}$. Hence ${}_B[\mathscr{F}(M), N] \cong {}_A[M, \mathscr{G}(N)]$ is a natural isomorphism.

The proof of part *b)* of the theorem is essentially dual to the proof of part *a)* and is left to the reader.

4.4. **Corollary:** *Let $\mathscr{F} : {}_A\mathscr{C} \to {}_B\mathscr{C}$ and $\mathscr{G} : {}_B\mathscr{C} \to {}_A\mathscr{C}$ be functors such that there is a natural isomorphism ${}_B[\mathscr{F}(M), N] \cong {}_A[M, \mathscr{G}(N)]$. Then there is an A-coflat object $Q \in {}_B\mathscr{C}_A$ (unique up to a isomorphism) such that*

$$\mathscr{F}(M) \cong Q \otimes_A M \quad \textit{and} \quad \mathscr{G}(N) \cong {}_B[Q, N]$$

natural in M resp. N.

Proof: Since $\mathscr{F}$ is a $\mathscr{C}$-functor and preserves colimits, Theorem 4.2. implies $\mathscr{F}(M) \cong Q \otimes_A M$. By Proposition 3.11 we also get $\mathscr{G}(N) \cong {}_B[Q, N]$.

If $\mathscr{C} = \mathbf{Z}$-Mod we know in the situation of Theorem 4.3 that $\mathscr{F}$ is an additive functor. This holds more generally.

4.5. **Proposition:** *Let $\mathscr{C} = K$-*Mod *with the usual tensor-product. Let $\mathscr{F} : {}_A\mathscr{C} \to {}_B\mathscr{C}$ be a $\mathscr{C}$-functor. Then $\mathscr{F}$ is K-additive, i.e. for all $f, g \in {}_A\mathscr{C}(M, N)$, $\varkappa \in K$ we have*

$$\mathscr{F}(f+g) = \mathscr{F}(f) + \mathscr{F}(g) \quad \textit{and} \quad \mathscr{F}(\varkappa f) = \varkappa \mathscr{F}(f).$$

Proof. First show that there is a natural isomorphism $\mathscr{F}(M \oplus M) \cong \mathscr{F}(M) \oplus \mathscr{F}(M)$ (natural in $M \in {}_A\mathscr{C}$) such that

$$(*) \qquad \begin{tikzcd} & \mathscr{F}(M \oplus M) \arrow[d, phantom, "\cong" rotate=90] \\ \mathscr{F}(M) \arrow[ur, "\mathscr{F}(q_i)"] \arrow[dr, "q_i"'] & \\ & \mathscr{F}(M) \oplus \mathscr{F}(M) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{F}(M \oplus M) \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, "\mathscr{F}(p_i)"] & \\ & \mathscr{F}(M) \\ \mathscr{F}(M) \oplus \mathscr{F}(M) \arrow[ur, "p_i"'] & \end{tikzcd}$$

for $i = 1, 2$ commute, where q_i are the i-th injections and p_i are the i-th projections of the direct sum. The isomorphism is given by $\mathscr{F}(M \oplus M) \cong \mathscr{F}(M \otimes (K \oplus K)) \cong \cong \mathscr{F}(M) \otimes (K \oplus K) \cong \mathscr{F}(M) \oplus \mathscr{F}(M)$. It is clearly natural in $M \in {}_A\mathscr{C}$. The commutativity of the first diagram follows from

$$\begin{tikzcd} \mathscr{F}(M) \arrow[rrr, "id_{\mathscr{F}(M)}"] \arrow[dr, "\cong"] \arrow[dd, "\mathscr{F}(q_i)"'] & & & \mathscr{F}(M) \arrow[dd, "q_i"] \\ & \mathscr{F}(M \otimes K) \arrow[r, phantom, "\cong"] \arrow[d, "\mathscr{F}(M \otimes q_i)"] & \mathscr{F}(M) \otimes K \arrow[ur, "\cong"] \arrow[d, "\mathscr{F}(M) \otimes q_i"] & \\ \mathscr{F}(M \oplus M) \arrow[r, phantom, "\cong"] & \mathscr{F}(M \otimes (K \oplus K)) \arrow[r, phantom, "\cong"] & \mathscr{F}(M) \otimes (K \oplus K) \arrow[r, phantom, "\cong"] & \mathscr{F}(M) \oplus \mathscr{F}(M) \end{tikzcd}$$

The other diagram follows similarly.

The diagrams $(*)$ imply immediately the commutativity of

$$\begin{tikzcd} & \mathscr{F}(M) \arrow[dl, "\mathscr{F}(\Delta)"'] \arrow[dr, "\Delta"] & \\ \mathscr{F}(M \oplus M) \arrow[rr, phantom, "\cong"] & & \mathscr{F}(M) \oplus \mathscr{F}(M) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{F}(M \oplus M) \arrow[rr, phantom, "\cong"] \arrow[dr, "\mathscr{F}(\nabla)"'] & & \mathscr{F}(M) \oplus \mathscr{F}(M) \arrow[dl, "\nabla"] \\ & \mathscr{F}(M) & \end{tikzcd}$$

Furthermore they imply the commutativity of

$$\begin{array}{ccc}
\mathscr{F}(M\oplus M) & \xrightarrow{\mathscr{F}(f\oplus g)} & \mathscr{F}(N\oplus N) \\
\wr\Vert & & \wr\Vert \\
\mathscr{F}(M)\oplus\mathscr{F}(M) & \xrightarrow{\mathscr{F}(f)\oplus\mathscr{F}(g)} & \mathscr{F}(N)\oplus\mathscr{F}(N).
\end{array}$$

Hence

$$\begin{tikzcd}
\mathscr{F}(M) \arrow[rrr] \arrow[dr, "\mathscr{F}(\Delta)"] \arrow[dd, "\Delta"'] & & & \mathscr{F}(N) \\
& \mathscr{F}(M\oplus M) \arrow[r, "\mathscr{F}(f\oplus g)"] \arrow[dl, phantom, "\cong"] & \mathscr{F}(N\oplus N) \arrow[ur, "\mathscr{F}(\nabla)"] \arrow[dr, phantom, "\cong"] & \\
\mathscr{F}(M)\oplus\mathscr{F}(M) \arrow[rrr, "\mathscr{F}(f)\oplus\mathscr{F}(g)"] & & & \mathscr{F}(N)\oplus\mathscr{F}(N) \arrow[uu, "\nabla"']
\end{tikzcd}$$

commutes, where the first horizontal arrow is $\mathscr{F}(f+g)=\mathscr{F}(f)+\mathscr{F}(g)$.

To show that $\mathscr{F}$ is K-linear, observe that for $f\in{}_A\mathscr{C}(M, N)$ and $\varkappa\in K$ the diagram

$$\begin{array}{ccc}
M\otimes K & \xrightarrow{f\otimes\varkappa} & N\otimes K \\
\wr\Vert & & \wr\Vert \\
M & \xrightarrow{\quad\varkappa\quad} & N
\end{array}$$

commutes. Hence the diagram

$$\begin{tikzcd}
\mathscr{F}(M)\otimes K \arrow[rrr, "\mathscr{F}(f)\otimes\varkappa"] \arrow[dr, phantom, "\cong"] \arrow[dd, phantom, "\wr\Vert"] & & & \mathscr{F}(N)\otimes K \arrow[dl, phantom, "\cong"] \arrow[dd, phantom, "\wr\Vert"] \\
& \mathscr{F}(M\otimes K) \arrow[r, "\mathscr{F}(f\otimes\varkappa)"] \arrow[dl, phantom, "\cong"] & \mathscr{F}(N\otimes K) \arrow[dr, phantom, "\cong"] & \\
\mathscr{F}(M) \arrow[rrr] & & & \mathscr{F}(N)
\end{tikzcd}$$

commutes, where the lower horizontal arrow is $\mathscr{F}(\varkappa f)=\varkappa\mathscr{F}(f)$.

Bibliography

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(Received November 20, 1975.)