

A sufficient condition for univalence

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In 1941 S. OZAKI [3] proved the following:

Theorem (OZAKI) Let $f(z) = z + a_2 z^2 + \dots$ be regular for $|z| < 1$. If $f(z)$ satisfies in $|z| < 1$ the condition

$$(1) \quad \operatorname{Re} \left[1 + z \frac{f''(z)}{f'(z)} \right] > -\frac{1}{2},$$

then $f(z)$ is univalent in $|z| < 1$.

T. UMEZAWA [4] in 1952 proved that (1) is also sufficient for the convexity of $f(z)$ in one direction.

In the present short note, besides generalizing Ozaki's theorem, we prove that the condition (1) implies that $f(z)$ is close-to-convex of order $1/2$.

Theorem 1. Let $f(z) = z + a_2 z^2 + \dots$ be regular for $|z| < 1$. If for any two real numbers λ and μ ; $\lambda > 0$, $f(z)$ satisfies in $|z| < 1$ condition:

$$(2) \quad \operatorname{Re} \left[1 + z \frac{f''(z)}{f'(z)} - (1 - \lambda - i\mu) z \frac{f'(z)}{f(z)} \right] > \begin{cases} -\frac{\lambda}{2}, & 0 < \lambda \leq 1 \\ -\frac{1}{2\lambda}, & \lambda > 1, \end{cases}$$

then $f(z)$ is univalent in $|z| < 1$.

PROOF. Let us put

$$A(z) = 1 + \frac{zf''(z)}{f'(z)} - (1 - \lambda - i\mu) \frac{zf'(z)}{f(z)}$$

and first consider the case when $0 < \lambda \leq 1$. From the first inequalities in (2) we then conclude that the function:

$$\frac{A(z) + \frac{\lambda}{2} - i \operatorname{Im} A(0)}{\operatorname{Re} A(0) + \frac{\lambda}{2}}$$

is regular in $|z| < 1$, assumes the value 1 at the origin, and has positive real part in $|z| < 1$. Using the expression for $A(z)$, we can therefore write

$$(3) \quad \frac{\lambda}{2} + 1 + \frac{zf''(z)}{f'(z)} - (1 - \lambda - i\mu) \frac{zf'(z)}{f(z)} = \frac{3}{2} \lambda P(z) + i\mu,$$

where $P(z) = 1 + p_1 z + p_2 z^2 + \dots$ is regular in $|z| < 1$ and has positive real part therein.

If in the right side of (3) for the function $P(z)$, we use the following well known Herglotz' representation:

$$P(z) = \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where $\mu(t)$ is a non-decreasing function defined for $-\pi \leq t \leq \pi$, with total variation equal to 1, that is,

$$\int_{-\pi}^{\pi} d\mu(t) = 1,$$

we obtain, after an integration, the following relation:

$$(4) \quad \log \left\{ \frac{z^{1-\lambda-i\mu} f'(z)}{f(z)^{1-\lambda-i\mu}} \right\} = -3\lambda \int_{-\pi}^{\pi} \log(1 - ze^{-it}) d\mu(t).$$

(Here principal values are understood to have been taken)

We now define a function $g(z)$ as follows:

$$(5) \quad \lambda \log \frac{g(z)}{z} = -2\lambda \int_{-\pi}^{\pi} \log(1 - ze^{-it}) d\mu(t).$$

It is readily seen that the function $g(z)$ is starlike in $|z| < 1$.

From (4) and (5) we obtain

$$(6) \quad \log \left\{ \frac{z^{1-i\mu} f'(z)}{f(z)^{1-\lambda-i\mu} g(z)^{\lambda}} \right\} = -\lambda \int_{-\pi}^{\pi} \log(1 - ze^{-it}) d\mu(t).$$

It is a simple matter to verify that the function $\log(1 - ze^{-it})$ maps $|z| < 1$ onto a convex domain and therefore we conclude that $-\lambda \int_{-\pi}^{\pi} \log(1 - ze^{-it}) d\mu(t)$ is subordinate to $\log(1 - ze^{-it})^{-\lambda}$ in $|z| < 1$ (For the definition of subordination, see [2; p 227]). In symbols

$$(7) \quad -\lambda \int_{-\pi}^{\pi} \log(1 - ze^{-it}) d\mu(t) < \log \frac{1}{(1 - ze^{-it})^{\lambda}}, \quad |z| < 1.$$

From (6) and (7) we then have

$$(8) \quad \frac{z^{1-i\mu} f'(z)}{f(z)^{1-\lambda-i\mu} g(z)^{\lambda}} < \frac{1}{(1 - ze^{-it})^{\lambda}} \quad \text{in } |z| < 1.$$

Since $0 < \lambda \leq 1$, from (8) we deduce that

$$\operatorname{Re} \left[\frac{z^{1-i\mu} f'(z)}{f(z)^{1-\lambda-i\mu} g(z)^\lambda} \right] > 0$$

in $|z| < 1$ and therefore $f(z)$ is BAZILEVIČ [1] and consequently univalent in $|z| < 1$. The case $\lambda > 1$ may be similarly handled. This completes the proof of Theorem 1.

If we take $\lambda = 1$ and $\mu = 0$ in Theorem 1, we obtain Ozaki's theorem.

Theorem 2. *If $f(z)$ satisfies (1) in $|z| < 1$, then $f(z)$ is close-to-convex of order $1/2$ in $|z| < 1$.*

PROOF. Taking $\lambda = 1$ and $\mu = 0$ in (8) we find that

$$\frac{zf'(z)}{g(z)} < \frac{1}{1-z^{-it}} \quad (|z| < 1)$$

from which we deduce that $\operatorname{Re} \{zf'(z)|g(z)\} > 1/2$ in $|z| < 1$ and therefore $f(z)$ is close-to-convex of order $1/2$ in $|z| < 1$.

References

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