

On Cauchy's nucleus

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Let G and H be arbitrary groups (written additively) and $f: G \rightarrow H$ an arbitrary function. Let us denote

$$N_f = \{x \in G: \forall_{y \in G} (f(x+y) = f(x) + f(y))\}.$$

The set N_f will be called a *Cauchy nucleus*. This set is empty or it is a subgroup of G (cf. [1] Lemma 2 p. 313).¹⁾

The following problem has been put forward at the symposium on Functional Equations in Debrecen in 1973.

Does there exist for every groups G, H and every subgroup S of the group G a function $f: G \rightarrow H$ such that $N_f = S$?²⁾

We are going to solve this problem in the present paper.

Theorem 1. *Let G be an arbitrary group and H an arbitrary group such that $2m \neq 0$ for some $m \in H$. Then for every subgroup S of the group G there exists a function $f: G \rightarrow H$ such that $N_f = S$.*

PROOF. Let us put

$$f(x) = \begin{cases} 0 & \text{for } x \in S, \\ m & \text{for } x \in G \setminus S. \end{cases}$$

We have

$$f(x+y) = 0 = f(x) + f(y) \quad \text{for } x, y \in S$$

and

$$f(x+y) = m = 0 + m = f(x) + f(y) \quad \text{for } x \in S, y \in G \setminus S.$$

Thus $S \subset N_f$.

Let us consider an arbitrary $x \in G \setminus S$. Then $-x \in G \setminus S$ and consequently, from the definition of the function f , we obtain

$$0 = f(0) = f(x-x) \neq f(x) + f(-x) = m + m = 2m.$$

It means that $x \notin N_f$. Hence $S = N_f$, which completes the proof.

Theorem 2. *Let G be an arbitrary group and S its subgroup such that for every $x \in G \setminus S$ there exists $y \in G \setminus S$ such that $x+y \in G \setminus S$. Let H be an arbitrary non-*

¹⁾ It has been proved in [1] p. 313 that the set $\{y \in G: \forall_{x \in G} (f(x+y) = f(x) + f(y))\}$ is empty or it is a subgroup of G . Proof for the set N_f can be realized in the similar way.

²⁾ Genuinely, the problem has been formulated for $G=H$.

trivial group, i.e. such that $\text{card } H > 1$. Then there exists a function $f: G \rightarrow H$ such that $N_f = S$.

PROOF. There exists an element $a \in H$ such that $a \neq 0$. We put

$$f(x) = \begin{cases} 0 & \text{for } x \in S, \\ a & \text{for } x \in G \setminus S. \end{cases}$$

We have

$$f(x+y) = 0 = f(x) + f(y) \quad \text{for } x \in S, y \in S.$$

and

$$f(x+y) = a = 0 + a = f(x) + f(y) \quad \text{for } x \in S, y \in G \setminus S.$$

This proves that $S \subset N_f$. If $G = S$, then the proof is complete. Suppose that $S \neq G$. Let us consider an arbitrary element $x \in G \setminus S$. By the assumption there exists $y \in G \setminus S$ such that $x+y \in G \setminus S$. Consequently, we have

$$f(x+y) = a \neq a + a = f(x) + f(y).$$

Thus $x \notin N_f$. It proves that $N_f = S$, which completes the proof.

The assumption on the subgroup S in Theorem 2 can be replaced by a better known condition. We shall show it in the following

Theorem 3. Let G be an arbitrary group and S its arbitrary subgroup. The following statements are equivalent:

- (A) For every $x \in G \setminus S$ there exists $y \in G \setminus S$ such that $x+y \in G \setminus S$,
- (B) The index of the subgroup S in the group G is different from two.

PROOF. If index S equals two, then $x+y \in S$ for every $x \in G \setminus S, y \in G \setminus S$. It proves that condition (A) implies condition (B). Let us suppose for the converse implication that condition (A) does not hold. It means that there exists $x_0 \in G \setminus S$ such that $x_0 + y \in S$ for every $y \in G \setminus S$. But if $y \in G \setminus S$, then also $-y \in G \setminus S$. Consequently, we have for every $y \in G \setminus S$: $x_0 - y \in S$, i.e. $y \in S + x_0$. It proves that $G \setminus S$ is a right coset of G with respect to subgroup S . Hence the index of S equals two, which completes the proof.

Theorems 1 and 2 do not solve the considered problem only in the case when index S equals two and $2x=0$ for every $x \in H$. In this case the answer to the set question may be negative.

In the next theorem we shall use the following result proved in [2] (Corollary 1).

Lemma 1. (cf. [2] Corollary 1). Let G, H be arbitrary groups and let S be a subgroup of G . The general solution of the functional equation³⁾

$$(1) \quad f(x+y) = f(x) + f(y) \quad \text{for } x \in S, y \in G,$$

where $f: G \rightarrow H$ is the looked for function, has the following form:

$$(2) \quad f(x) = g(x-u) + h(u) \quad \text{for } x \in S+U, u \in U,$$

where

³⁾ The following functional equation has been solved in note [2] $f(x \oplus y) = f(x) \oplus f(y)$ for $x \in G, y \in S$. Equation (1) can be reduced to this equation by putting $y \oplus x = x + y$.

$g: S \rightarrow H$ is an arbitrary homomorphism, $U \subset G$ is an arbitrary selector of the family $\{S+x\}_{x \in G}$ i.e. $U \cap \{S+x\}$ is a one-element set for every $x \in G$,
 $h: U \rightarrow H$ is an arbitrary function satisfying condition

$$(3) \quad h(u_0) = g(u_0) \quad \text{for} \quad \{u_0\} = U \cap S.$$

Lemma 2. Let H be a group. If $2x=0$ for every $x \in H$ then the group H is abelian. The proof is obvious.

Theorem 4. Let G, H be groups and let $2m=0$ for every $m \in H$. Let S be a subgroup of G of the index two. Then the following statements are equivalent:

(C) There exists a function $f: G \rightarrow H$ such that $N_f = S$,
 (D) There exists a normal subgroup T of the group S such that the group S/T is isomorphic to some subgroup H and for every $x \in G \setminus S$ there exists $y \in G \setminus S$ such that

$$(4) \quad x+y-x-y \notin T,$$

or there exist $y \in G \setminus S$, $u \in G \setminus S$ such that

$$(5) \quad x-u+y-u-x \notin T.$$

PROOF. Let us consider an arbitrary solution of the functional equation (1), i.e. a function $f: G \rightarrow H$ of the form (2), where U, u_0, u have the meanings as in Lemma 1. We have for $x \in S$:

$$f(x) = g(x-u_0) + h(u_0) = g(x) - g(u_0) + h(u_0) = g(x).$$

Since the index of S equals two, function f may be written as follows

$$(6) \quad f(x) = \begin{cases} g(x) & \text{for } x \in S, \\ g(x-u) + h(u) & \text{for } x \in G \setminus S, \end{cases} \quad \{u\} = (G \setminus S) \cap U.$$

Suppose that condition (C) holds. It means that there exists a solution f of equation (1) such that for every $x \in G \setminus S$ there exists $y \in G$ such that

$$(7) \quad f(x+y) \neq f(x) + f(y).$$

Two cases may occur: 1) $y \in S$, 2) $y \in G \setminus S$. In case 1)

$$x+y \in G \setminus S$$

and hence

$$f(x+y) = g(x+y-u) + h(u).$$

This equality implies that inequality (6) may be written in the following way:

$$g(x+y-u) + h(u) \neq g(x-u) + h(u) + g(y).$$

In virtue of Lemma 2 H is abelian. In consequence, the last inequality is equivalent to the following one

$$(8) \quad g(x+y-u) \neq g(y) + g(x-u) = g(y+x-u).$$

In case 1) $x+y \in S$ and in consequence inequality (7) receives the form

$$g(x+y) \neq g(x-u) + h(u) + g(y-u) + h(u).$$

But H is abelian and $2m=0$ for $m \in H$. Hence, the last inequality may be written in an equivalent form as follows

$$(9) \quad g(x+y) \neq g(x-u) + g(y-u) = g(x-u+y-u).$$

Let us put $T = \text{Ker } g$. Obviously, S/T is isomorphic to the subgroup $g(S)$ of the group H . Inequality (8) is equivalent to the condition $x+y-u-(y+x-u) \notin T$ and hence it is equivalent to condition (4). Inequality (9) is equivalent to the condition (4). Inequality (9) is equivalent to the condition $x-u+y-u-(x+y) \notin T$ i.e. to condition (5). Thus, we have proved that if condition (C) is satisfied, so is condition (D).

Suppose now that condition (D) is fulfilled. Let t denote the isomorphism of the group S/T onto some subgroup of the group H and let $k: S \rightarrow S/T$ be the canonical mapping. We put $g = t \circ k$. Let U be any selector of the family $\{S, G \setminus S\}$ such that $\{u\} = (G \setminus S) \cap U$, where u is the element occurring in (5). Let $h: U \rightarrow H$ be an arbitrary function satisfying (3). Let us consider function f defined by (6). It is immediately seen that $T = \text{Ker } g$. We have proved in the demonstration of the implication (C) $\Rightarrow$ (D) that the alternative of conditions (4) and (5), under respective notations, is equivalent to inequality (7). Hence, for every $x \in G \setminus S$ there exists $y \in G$ such that (7) holds. Because, in addition f satisfies (1) it proves that $N_f = S$, which completes the proof.

We obtain from Theorem 4 the following

Corollary. Let all assumptions of Theorem 4 be fulfilled and let additionally $2x=0$ for every $x \in G$. Then there does not exist a function $f: G \rightarrow H$ such that $N_f = S$.

PROOF. Let us observe that under our assumptions for every $x, y, u \in G$ and every subgroup T of the group S conditions (4) and (5) are not valid. This statement and Theorem 4 completes the proof.

We shall illustrate our considerations by the following

Example. Let G be the additive group of integers and $S \subset G$ a subgroup assembled from all even integers. As the group H we take the group $Z_2 = \{0, 1\}$ with addition mod 2. Let $T \subset S$ be a subgroup of integers of the form $4n$ for n runs over the set of integers. It is obvious that the group S/T is isomorphic to the group H . Let us put $y=u=1$. Then $y, u \in G \setminus S$ and for every $x \in G$ $x-u+y-u-y-x = -2 \notin T$. It means that condition (D) of Theorem 4 is satisfied. Hence, there exists a function $f: G \rightarrow H$ such that $N_f = S$. The construction of this function is given in the last part of the proof of Theorem 4 and in Lemma 1. In the considered case the function g occurring in this construction has the form

$$g(x) = \begin{cases} 0 & \text{for } x = 4n, \quad n \in Z, \\ 1 & \text{for } x = 4n+2, \quad n \in Z. \end{cases}$$

In consequence, if we put $h(1)=0$ then we obtain (by this construction) the function f defined as follows

$$f(x) = \begin{cases} 0 & \text{for } x = 4n, \quad n \in Z, \\ 1 & \text{for } x = 4n+2, \quad n \in Z, \\ 0 & \text{for } x = 4n+1, \quad n \in Z, \\ 1 & \text{for } x = 4n+3, \quad n \in Z. \end{cases}$$

It can be also verified that for groups G, H, S there exist only two functions $f: G \rightarrow H$ such that $N_f = S$

References

- [1] M. KUCZMA, Cauchy's functional equation on a restricted domain, *Coll. Math.* **28** (1973), 313—315.
- [2] J. TABOR, Solution of Cauchy's functional equation on a restricted domain, *Coll. Math.* (to appear).

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