

On the convolution of distributions

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1. Introduction.

The definition of the convolution of two distributions was extended recently in a paper by JONES [2] to cover certain pairs of distributions which could not be convolved in the sense of the previous definition. The convolution $f * g$ of two distributions f and g was defined as the limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that

$$\lim_{n \rightarrow \infty} \int \varphi(x) (f_n * g_n)(x) dx = \int \varphi(x) h(x) dx$$

for all fine functions φ in the terminology of JONES [3], where

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

and τ is an infinitely differentiable function satisfying the following conditions:

- (1) $\tau(x) = \tau(-x),$
- (2) $0 \leq \tau(x) \leq 1,$
- (3) $\tau(x) = 1 \quad \text{for} \quad |x| \leq \frac{1}{2},$
- (4) $\tau(x) = 0 \quad \text{for} \quad |x| \geq 1.$

It was proved in [2] that with this definition

$$1 * \operatorname{sgn} x = x$$

and in [1] the generalization

$$(1) \quad x^r * (\operatorname{sgn} x \cdot x^r) = \frac{(r!)}{(2r+1)!} x^{2r+1}$$

was proved, for $r=0, 1, 2, \dots$.

In the following we prove a theorem which gives equation (1) as a particular case.

2. Theorem. *Let f and g be arbitrary distributions for which*

$$f(x) = 0 \quad \text{for} \quad x \leq a,$$

$$g(x) = 0 \quad \text{for} \quad x \geq b,$$

where a, b are arbitrary real numbers. Then the convolution $(f+g)*(f-g)$ exists and

$$(f+g)*(f-g) = f*f - g*g.$$

PROOF. We note first of all that the convolutions $f*f$ and $g*g$ exist without the use of Jones' definition of the convolution.

We write

$$(f+g)_n(x) = (f+g)(x)\tau(x/n),$$

$$(f-g)_n(x) = (f-g)(x)\tau(x/n).$$

It follows that the convolution $(f+g)_n*(f-g)_n$ exists and

$$(f+g)_n*(f-g)_n = (f_n+g_n)*(f_n-g_n) = f_n*f_n + g_n*f_n - f_n*g_n - g_n*g_n.$$

Since

$$g_n*f_n = f_n*g_n$$

and

$$\lim_{n \rightarrow \infty} f_n*f_n = f*f, \quad \lim_{n \rightarrow \infty} g_n*g_n = g*g$$

it follows that

$$\lim_{n \rightarrow \infty} (f+g)_n*(f-g)_n = f*f - g*g,$$

completing the proof of the theorem.

An obvious extension of this theorem is that if h_1 and h_2 are arbitrary distributions with compact support and f and g are distributions satisfying the conditions of the theorem, then the convolution $(f+g+h_1)*(f-g+h_2)$ exists and

$$(f+g+h_1)*(f-g+h_2) = f*f - g*g + h_1*(f-g) + h_2*(f+g),$$

the convolutions $h_1*(f-g)$ and $h_2*(f+g)$ existing since h_1 and h_2 have compact support.

Since

$$x_+^\alpha * x_+^\beta = B(\alpha+1, \beta+1)x_+^{\alpha+\beta+1},$$

$$x_-^\alpha * x_-^\beta = B(\alpha+1, \beta+1)x_-^{\alpha+\beta+1},$$

for $\alpha, \beta, \alpha+\beta+1 \neq -1, -2, \dots$, where B denotes the beta function, it follows from the theorem that

$$(x_+^\alpha + x_-^\beta)*(x_+^\alpha - x_-^\beta) = B(\alpha+1, \alpha+1)x_+^{2\alpha+1} - B(\beta+1, \beta+1)x_-^{2\beta+1}$$

for

$$\alpha, \beta \neq -1, -2, \dots,$$

and $(x_+^\alpha + x_-^\alpha)*(x_+^\alpha - x_-^\alpha) = |x|^\alpha * (\operatorname{sgn} x \cdot |x|^\alpha) = B(\alpha+1, \alpha+1) \operatorname{sgn} x \cdot |x|^{2\alpha+1}$

for $\alpha \neq -1, -2, \dots$, the particular case $\alpha=r=0, 1, 2, \dots$, being equation (1).

More generally it follows that the convolutions

$$(x_+^\alpha \ln^p x_+ + x_-^\beta \ln^q x_-)*(x_+^\alpha \ln^p x_+ - x_-^\beta \ln^q x_-)$$

and

$$(|x|^\alpha \ln^p |x|)*(\operatorname{sgn} x \cdot |x|^\alpha \ln^p |x|)$$

exist for all α, β and $p, q=0, 1, 2, \dots$.

References

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