

Some problems on horizontal and complete lifts of $\theta(4, -2)$ -structure

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Summary.

The horizontal and complete lifts from a differentiable manifold M of class c^∞ to its cotangent bundle $cT(M)$ have been studied by YANO and PATTERSON ([2], [3]). YANO, HOUH and CHEN [4] have studied the structures defined by a tensor field θ of type $(1, 1)$ satisfying $\theta^4 \pm \theta^2 = 0$. The purpose of the present paper is to obtain certain results on horizontal and complete lifts of $\theta(4, -2)$ -structure.

In section 2, we have obtained the conditions under which the complete lift of a tensor field θ of type $(1, 1)$ having a $\theta(4, -2)$ -structure in M may have a $\theta(4, -2)$ -structure in $cT(M)$. In section 3, we have proved that the processes of computing the Nijenhuis tensor of θ^2 and taking the complete lift are commutative. In section 4, we have proved that the horizontal lift of a tensor field θ of type $(1, 1)$ having a $\theta(4, -2)$ -structure in M will also have a $\theta(4, -2)$ -structure in $cT(M)$.

1. Preliminaries.

Let M be a differentiable manifold of class c^∞ and dimension n , and let $cT(M)$ be the cotangent bundle of M . Then $cT(M)$ is also a differentiable manifold of class c^∞ ; the dimension of $cT(M)$ is $2n$.

Throughout this paper, we use the following notations and conventions:

- (i) $\pi: cT(M) \rightarrow M$ is the projection of $cT(M)$ onto M .
- (ii) Suffixes $a, b, c, \dots, h, i, j, \dots$ take the values 1 to n and $\bar{i} = i + n$, etc. Suffixes $A, B, C, D \dots$ take the values 1 to $2n$.
- (iii) $\mathcal{J}_s^r(M)$ denotes the set of tensor fields of class c^∞ and type (r, s) in M , and $\mathcal{J}_s^r(cT(M))$ denotes the corresponding set of tensor fields in $cT(M)$.
- (iv) Vector fields in M are denoted by X, Y, Z . The Lie product of X and Y is denoted by $[X, Y]$ and the Lie derivative with respect to X is denoted by $\mathcal{L}_X$.

If A is a point in M , then $\pi^{-1}(A)$ is the fibre over A . Any point $P \in \pi^{-1}(A)$ is an ordered pair (A, p_A) , where p is a 1-form in M and p_A is its value at A . Let U be a coordinate neighbourhood in M such that $A \in U$. Then U induces a coordinate neighbourhood $\pi^{-1}(U)$ in $cT(M)$ and $P \in \pi^{-1}(U)$.

If $X, Y, Z \in \mathcal{J}_0^1(M)$ and $\theta \in \mathcal{J}_1^1(M)$. Then we have [2]

$$(1.1) \quad (X+Y)^c = X^c + Y^c,$$

$$(1.2) \quad \theta^c(Z^c) = (\theta Z)^c + (\mathcal{L}_Z \theta)^c,$$

where X^c, Y^c, Z^c are the complete lifts of the vector fields X, Y, Z respectively and θ^c is the complete lift of the tensor field θ .

2. The complete lift of a $\Theta(4, -2)$ -structure.

Let Θ be a tensor field of type $(1, 1)$ and of class c^∞ in M which implies that $\Theta \in \mathcal{T}_1^1(M)$ and let Θ satisfy the following:

$$(2.1) \quad \Theta^4 - \Theta^2 = 0.$$

That is, Θ is a $\Theta(4, -2)$ -structure on M [4].

Let Θ_i^h be the components of Θ at a point A in a coordinate neighbourhood U . Then the complete lift Θ^C of Θ is also a tensor field of type $(1, 1)$ in $cT(M)$ whose components $\tilde{\Theta}_B^A$ in $\pi^{-1}(U)$ are given by [2]

$$(2.2) \quad \tilde{\Theta}_i^h = \Theta_i^h, \quad \tilde{\Theta}_i^h = 0,$$

$$\tilde{\Theta}_i^h = p_a \left(\frac{\partial \Theta_h^a}{\partial x^i} - \frac{\partial \Theta_i^a}{\partial x^h} \right), \quad \tilde{\Theta}_i^h = \Theta_h^i,$$

where $(x^1, x^2, \dots, x^n)$ are coordinates of A relative to U and p_A has components $(p_1, p_2, \dots, p_n)$. Therefore, we have

$$(2.3) \quad \Theta^C \stackrel{\text{def}}{=} \tilde{\Theta}_i^h = \begin{pmatrix} \Theta_i^h & 0 \\ p_a(\partial_i \Theta_h^a - \partial_h \Theta_i^a) & \Theta_h^i \end{pmatrix},$$

where $\partial_i \equiv \partial/\partial x^i$.

Now, in consequence of (2.3), we have

$$(2.4) \quad \begin{aligned} \tilde{\Theta}_i^h \tilde{\Theta}_j^i &= \begin{pmatrix} \Theta_i^h & 0 \\ p_a(\partial_i \Theta_h^a - \partial_h \Theta_i^a) & \Theta_h^i \end{pmatrix} \begin{pmatrix} \Theta_j^i & 0 \\ p_t(\partial_j \Theta_i^t - \partial_i \Theta_j^t) & \Theta_i^t \end{pmatrix}, \\ &= \begin{pmatrix} \Theta_i^h \Theta_j^i & 0 \\ 2\Theta_j^i p_a \partial_{[i} \Theta_{h]}^a + 2\Theta_h^i p_t \partial_{[j} \Theta_{i]}^t & \Theta_h^i \Theta_j^i \end{pmatrix}, \end{aligned}$$

where $2\partial_{[i} \Theta_{h]}^a \stackrel{\text{def}}{=}} (\partial_i \Theta_h^a - \partial_h \Theta_i^a)$.

Let us put

$$(2.5) \quad L_{hj} \stackrel{\text{def}}{=} 2(\Theta_j^i p_a \partial_{[i} \Theta_{h]}^a + \Theta_h^i p_t \partial_{[j} \Theta_{i]}^t).$$

Then in view of (2.5), equation (2.4) becomes

$$(2.6) \quad \tilde{\Theta}_i^h \tilde{\Theta}_j^i = \begin{pmatrix} \Theta_i^h \Theta_j^i & 0 \\ L_{hj} & \Theta_h^i \Theta_j^i \end{pmatrix}.$$

Hence we have

$$(2.7) \quad \begin{aligned} \tilde{\Theta}_i^h \tilde{\Theta}_j^i \tilde{\Theta}_k^j \tilde{\Theta}_l^k &= \begin{pmatrix} \Theta_i^h \Theta_j^i & 0 \\ L_{hj} & \Theta_h^i \Theta_j^i \end{pmatrix} \begin{pmatrix} \Theta_k^j \Theta_l^k & 0 \\ L_{jl} & \Theta_j^k \Theta_l^k \end{pmatrix}, \\ &= \begin{pmatrix} \Theta_i^h \Theta_j^i \Theta_k^j \Theta_l^k & 0 \\ L_{hj} \Theta_k^j \Theta_l^k + \Theta_h^i \Theta_j^i L_{jl} & \Theta_h^i \Theta_j^i \Theta_k^j \Theta_l^k \end{pmatrix}. \end{aligned}$$

Now we shall prove the following lemma and theorems:

Lemma (2.1). *In order that the complete lift Θ^C in $cT(M)$ of a tensor field Θ having $\Theta(4, -2)$ -structure in M may have $\Theta(4, -2)$ -structure in $cT(M)$, it is necessary and sufficient that*

$$(2.8) \quad L_{hj} \Theta_k^j \Theta_l^k + \Theta_h^i \Theta_l^j L_{ji} = L_{hl}.$$

PROOF. Since Θ satisfies $\Theta^4 - \Theta^2 = 0$, therefore

$$(2.9) \quad \Theta_h^i \Theta_j^i \Theta_k^j \Theta_l^k = \Theta_r^h \Theta_l^r.$$

Thus in view of (2.7) and (2.9), we obtain

$$(2.10) \quad \tilde{\Theta}_i^h \tilde{\Theta}_j^i \tilde{\Theta}_k^j \tilde{\Theta}_l^k = \begin{pmatrix} \Theta_r^h \Theta_l^r & 0 \\ L_{hj} \Theta_k^j \Theta_l^k + \Theta_h^i \Theta_l^j L_{ji} & \Theta_r^h \Theta_l^r \end{pmatrix}.$$

Let us assume that the condition (2.8) is satisfied. Then from (2.10) we have

$$\tilde{\Theta}_i^h \tilde{\Theta}_j^i \tilde{\Theta}_k^j \tilde{\Theta}_l^k = \begin{pmatrix} \Theta_r^h \Theta_l^r & 0 \\ L_{hl} & \Theta_r^h \Theta_l^r \end{pmatrix},$$

which in view of (2.6) yields

$$\tilde{\Theta}_i^h \tilde{\Theta}_j^i \tilde{\Theta}_k^j \tilde{\Theta}_l^k = \tilde{\Theta}_r^h \tilde{\Theta}_l^r.$$

Thus Θ^C in $cT(M)$ satisfies

$$(\Theta^C)^4 - (\Theta^C)^2 = 0.$$

The necessary condition can be proved in a straight forward manner.

Theorem (2.1). *In order that the complete lift Θ^C in $cT(M)$ of a tensor field Θ having $\Theta(4, -2)$ -structure in M may have $\Theta(4, -2)$ -structure in $cT(M)$, it is necessary and sufficient that*

$$(2.11) \quad \begin{aligned} & \Theta_j^i \Theta_k^j \Theta_l^k p_a \partial_{[i} \Theta_{h]}^a + \Theta_h^i \Theta_k^j \Theta_l^k p_a \partial_{[j} \Theta_{i]}^a + \Theta_h^i \Theta_l^j \Theta_r^s p_r \partial_{[s} \Theta_{j]}^r + \\ & + \Theta_h^i \Theta_l^j \Theta_r^s p_r \partial_{[j]l} \Theta_{s]}^r - \Theta_l^i p_a \partial_{[i} \Theta_{h]}^a - \Theta_h^i p_a \partial_{[i} \Theta_{l]}^a = 0. \end{aligned}$$

PROOF. From lemma (2.1) it follows that the complete lift Θ^C in $cT(M)$ of Θ in M satisfying $\Theta^4 - \Theta^2 = 0$ will satisfy $(\Theta^C)^4 - (\Theta^C)^2 = 0$, if and only if (2.8) is satisfied. Now the condition (2.8) in view of (2.5) can be expressed as

$$\begin{aligned} & \{\Theta_j^i p_a \partial_{[i} \Theta_{h]}^a + \Theta_h^i p_a \partial_{[j} \Theta_{i]}^a\} \Theta_k^j \Theta_l^k + \Theta_h^i \Theta_l^j \{\Theta_r^s p_r \partial_{[s} \Theta_{j]}^r + \Theta_j^s p_r \partial_{[l} \Theta_{s]}^r\} = \\ & = \Theta_l^i p_a \partial_{[i} \Theta_{h]}^a + \Theta_h^i p_a \partial_{[l} \Theta_{i]}^a, \end{aligned}$$

or

$$(2.12) \quad \begin{aligned} & \Theta_j^i p_a \partial_{[i} \Theta_{h]}^a \Theta_k^j \Theta_l^k + \Theta_h^i p_a \partial_{[j} \Theta_{i]}^a \Theta_k^j \Theta_l^k + \Theta_h^i \Theta_l^j \Theta_r^s p_r \partial_{[s} \Theta_{j]}^r + \\ & + \Theta_h^i \Theta_l^j \Theta_r^s p_r \partial_{[l} \Theta_{s]}^r - \Theta_l^i p_a \partial_{[i} \Theta_{h]}^a - \Theta_h^i p_a \partial_{[l} \Theta_{i]}^a = 0. \end{aligned}$$

Thus the theorem follows.

Theorem (2.2). *In order that the complete lift Θ^C in $cT(M)$ of a tensor field Θ having $\Theta(4, -2)$ -structure in M may have $\Theta(4, -2)$ -structure in $cT(M)$, it is necessary and sufficient that*

$$(2.13) \quad (\text{curl } V + 2p_a \Gamma_{r[h}^a \Theta_{i]}^r)^4 - (\text{curl } V + 2p_a \Gamma_{r[h}^a \Theta_{i]}^r)^2 = 0,$$

where we have defined the following:

$$(2.14) \quad \text{curl } V = V_{h,i} - V_{i,h} \stackrel{\text{def}}{=} p_a(\Theta_{h,i}^a - \Theta_{i,h}^a)$$

and

$$(2.15) \quad 2\Gamma_{r[h}\Theta_{i]}^a \stackrel{\text{def}}{=} \Gamma_{rh}^a \Theta_i^r - \Gamma_{ri}^a \Theta_h^r.$$

PROOF. Let us assume that the complete lift Θ^C in $cT(M)$ of Θ having $\Theta(4, -2)$ -structure in M has a $\Theta(4, -2)$ -structure in $cT(M)$. Since in view of (2.2), the only relevant part of the components of Θ^C in $cT(M)$ is

$$p_a(\partial_i \Theta_h^a - \partial_h \Theta_i^a), \quad \text{where } \partial_i \equiv \partial/\partial x^i.$$

Therefore Θ^C satisfies

$$(2.16) \quad [p_a(\partial_i \Theta_h^a - \partial_h \Theta_i^a)]^4 - [p_a(\partial_i \Theta_h^a - \partial_h \Theta_i^a)]^2 = 0.$$

Let Γ_{jk}^i be a symmetric affine connection in M . Then the covariant derivative of Θ_h^a with respect to x^i is given by

$$(2.17) \quad \Theta_{h,i}^a = \partial_i \Theta_h^a + \Gamma_{ri}^a \Theta_h^r - \Gamma_{hi}^s \Theta_s^a.$$

In consequence of (2.17), equation (2.16) assumes the form

$$[p_a(\Theta_{h,i}^a - \Theta_{i,h}^a + \Gamma_{rh}^a \Theta_i^r - \Gamma_{ri}^a \Theta_h^r)]^4 - [p_a(\Theta_{h,i}^a - \Theta_{i,h}^a + \Gamma_{rh}^a \Theta_i^r - \Gamma_{ri}^a \Theta_h^r)]^2 = 0.$$

As regards a fixed, Θ is covariant. Hence in view of (2.15) we have

$$(2.18) \quad [p_a\{(\Theta_{h,i}^a - \Theta_{i,h}^a) + 2\Gamma_{r[h}\Theta_{i]}^r\}]^4 - [p_a\{(\Theta_{h,i}^a - \Theta_{i,h}^a) + 2\Gamma_{r[h}\Theta_{i]}^r\}]^2 = 0.$$

Equation (2.18) in consequence of (2.14) yields (2.13). Similarly, it can also be proved that the condition is sufficient.

Corollary (2.1). In a Kähler space with $\Theta(4, -2)$ -structure, the complete lift Θ^C of Θ has a $\Theta(4, -2)$ -structure in $cT(M)$ if and only if

$$(2.19) \quad (2p_a \Gamma_{r[h}\Theta_{i]}^a)^4 - (2p_a \Gamma_{r[h}\Theta_{i]}^a)^2 = 0.$$

PROOF. For a Kählerian space, we have [1]

$$(2.20) \quad \Theta_{h,i}^a = 0 \quad \text{or} \quad \text{curl } V = 0.$$

Thus by virtue of (2.13) and (2.20), the result follows.

Corollary (2.2). In a manifold M with $\Theta(4, -2)$ -structure if $\text{curl } V = 0$, then Θ^C has a $\Theta(4, -2)$ -structure in $cT(M)$ if and only if (2.19) is satisfied.

PROOF. The proof is similar to that of Corollary (2.1).

3. Nijenhuis tensor of the complete lift of Θ^4 .

Let $\Theta \in \mathcal{J}_1^1(M)$. Then the Nijenhuis tensor of Θ is given by [2]

$$(3.1) \quad N_{\Theta, \Theta}(X, Y) = [\Theta X, \Theta Y] - \Theta[\Theta X, Y] - \Theta[X, \Theta Y] + \Theta^2[X, Y].$$

For every $\Theta \in \mathcal{J}_1^1(M)$, we also have [2]

$$(3.2) \quad (\Theta^C)^2 = (\Theta^2)^C + (N_{\Theta, \Theta})^V.$$

Let Θ be a $\Theta(4, -2)$ -structure on M . That is, $\Theta^4 - \Theta^2 = 0$. Now we shall prove the following theorems:

Theorem (3.1). *The Nijenhuis tensor of the complete lift of Θ^4 vanishes if Lie derivatives of the tensor field Θ^2 with respect to X and Y are both zero and Θ is an almost product structure on M .*

PROOF. In consequence of (3.1), the Nijenhuis tensor of $(\Theta^4)^c$ is given by

$$N_{(\Theta^4)^c, (\Theta^4)^c}(X^c, Y^c) = [(\Theta^4)^c X^c, (\Theta^4)^c Y^c] - (\Theta^4)^c [(\Theta^4)^c X^c, Y^c] - (\Theta^4)^c [X^c, (\Theta^4)^c Y^c] + (\Theta^4)^c (\Theta^4)^c [X^c, Y^c],$$

which in view of (2.1) yields

$$(3.3) \quad N_{(\Theta^4)^c, (\Theta^4)^c}(X^c, Y^c) = [(\Theta^2)^c X^c, (\Theta^2)^c Y^c] - (\Theta^2)^c [(\Theta^2)^c X^c, Y^c] - (\Theta^2)^c [X^c, (\Theta^2)^c Y^c] + (\Theta^2)^c (\Theta^2)^c [X^c, Y^c].$$

In consequence of (1.2), we have

$$(3.4) \quad (\Theta^2)^c X^c = (\Theta^2 X)^c + (\mathcal{L}_X \Theta^2)^v.$$

Therefore by virtue of (3.4), equation (3.3) becomes

$$(3.5) \quad N_{(\Theta^4)^c, (\Theta^4)^c}(X^c, Y^c) = [(\Theta^2 X)^c, (\Theta^2 Y)^c] + [(\mathcal{L}_X \Theta^2)^v, (\Theta^2 Y)^c] + [(\Theta^2 X)^c, (\mathcal{L}_Y \Theta^2)^v] + [(\mathcal{L}_X \Theta^2)^v, (\mathcal{L}_Y \Theta^2)^v] - (\Theta^2)^c [(\Theta^2 X)^c, Y^c] - (\Theta^2)^c [(\mathcal{L}_X \Theta^2)^v, Y^c] - (\Theta^2)^c [X^c, (\Theta^2 X)^c] - (\Theta^2)^c [X^c, (\mathcal{L}_Y \Theta^2)^v] + (\Theta^2)^c (\Theta^2)^c [X^c, Y^c].$$

If Lie derivatives of the tensor field Θ^2 with respect to X and Y are both zero. Then we have

$$\mathcal{L}_X \Theta^2 = 0 \quad \text{and} \quad \mathcal{L}_Y \Theta^2 = 0.$$

Therefore equation (3.5) reduces to

$$(3.6) \quad N_{(\Theta^4)^c, (\Theta^4)^c}(X^c, Y^c) = [(\Theta^2 X)^c, (\Theta^2 Y)^c] - (\Theta^2)^c [(\Theta^2 X)^c, Y^c] - (\Theta^2)^c [X^c, (\Theta^2 Y)^c] + (\Theta^2)^c (\Theta^2)^c [X^c, Y^c].$$

Now for every $X, Y \in \mathcal{J}_0^1(M)$, we have [2]

$$(3.7) \quad [X^c, Y^c] = [X, Y]^c$$

Therefore by virtue of (3.7), equation (3.6) becomes

$$(3.8) \quad N_{(\Theta^4)^c, (\Theta^4)^c}(X^c, Y^c) = [\Theta^2 X, \Theta^2 Y]^c - (\Theta^2)^c [\Theta^2 X, Y]^c - (\Theta^2)^c [X, \Theta^2 Y]^c + (\Theta^2)^c (\Theta^2)^c [X, Y]^c.$$

If Θ is also an almost product structure on M , $\Theta^2 = I$, where I is the unit tensor field. Hence, equation (3.8) yields

$$N_{(\Theta^4)^c, (\Theta^4)^c}(X^c, Y^c) = 0.$$

Theorem (3.2). *The Nijenhuis tensor of the complete lift of Θ^4 is equal to the complete lift of the Nijenhuis tensor of Θ^2 if*

$$(i) \quad \mathcal{L}_X \Theta^2 = 0, \quad \mathcal{L}_Y \Theta^2 = 0$$

and

$$(ii) \quad [X, Y]^C = 0, \quad \tilde{K}^V = 0;$$

where

$$(3.9) \quad \tilde{K} \stackrel{\text{def}}{=} \mathcal{L}_{[\Theta^2 X, Y]} \Theta^2 + \mathcal{L}_{[X, \Theta^2 Y]} \Theta^2 - \mathcal{L}_{[X, Y]} \Theta^4.$$

PROOF. In consequence of (1.1) and (3.1), we have

$$(N_{\Theta^2, \Theta^2}(X, Y))^C = [\Theta^2 X, \Theta^2 Y]^C - (\Theta^2[\Theta^2 X, Y])^C - (\Theta^2[X, \Theta^2 Y])^C + (\Theta^4[X, Y])^C,$$

which in view of (3.4) yields

$$(3.10) \quad \begin{aligned} (N_{\Theta^2, \Theta^2}(X, Y))^C &= [\Theta^2 X, \Theta^2 Y]^C - \{(\Theta^2)^C[\Theta^2 X, Y]^C - (\mathcal{L}_{[\Theta^2 X, Y]} \Theta^2)^V\} - \\ &\quad - \{(\Theta^2)^C[X, \Theta^2 Y]^C - (\mathcal{L}_{[X, \Theta^2 Y]} \Theta^2)^V\} + \{(\Theta^4)^C[X, Y]^C - (\mathcal{L}_{[X, Y]} \Theta^4)^V\}. \end{aligned}$$

In consequence of (3.2), we have

$$(3.11) \quad (\Theta^2)^C(\Theta^2)^C = (\Theta^4)^C + (N_{\Theta^2, \Theta^2})^V.$$

Therefore by virtue of (3.11), equation (3.10) becomes

$$(3.12) \quad \begin{aligned} (N_{\Theta^2, \Theta^2}(X, Y))^C &= [\Theta^2 X, \Theta^2 Y]^C - (\Theta^2)^C[\Theta^2 X, Y]^C - (\Theta^2)^C[X, \Theta^2 Y]^C + \\ &\quad + (\Theta^2)^C(\Theta^2)^C[X, Y]^C - (N_{\Theta^2, \Theta^2})^V[X, Y]^C + (\mathcal{L}_{[\Theta^2 X, Y]} \Theta^2 + \mathcal{L}_{[X, \Theta^2 Y]} \Theta^2 - \mathcal{L}_{[X, Y]} \Theta^4)^V. \end{aligned}$$

Hence in view of (3.12), equation (3.8) yields

$$\begin{aligned} N_{(\Theta^4)^C, (\Theta^4)^C}(X^C, Y^C) &= (N_{\Theta^2, \Theta^2}(X, Y))^C + (N_{\Theta^2, \Theta^2})^V[X, Y]^C - \\ &\quad - (\mathcal{L}_{[\Theta^2 X, Y]} \Theta^2 + \mathcal{L}_{[X, \Theta^2 Y]} \Theta^2 - \mathcal{L}_{[X, Y]} \Theta^4)^V, \end{aligned}$$

which in consequence of (3.9) gives

$$(3.13) \quad N_{(\Theta^4)^C, (\Theta^4)^C}(X^C, Y^C) = (N_{\Theta^2, \Theta^2}(X, Y))^C + (N_{\Theta^2, \Theta^2})^V[X, Y]^C - \tilde{K}^V.$$

If $[X, Y]^C = 0$ and $\tilde{K}^V = 0$, then equation (3.13) reduces to

$$N_{(\Theta^4)^C, (\Theta^4)^C}(X^C, Y^C) = (N_{\Theta^2, \Theta^2}(X, Y))^C.$$

Theorem (3.3). *The Nijenhuis tensor of the complete lift of Θ^4 is equal to the complete lift of the Nijenhuis tensor of Θ^2 if*

$$(i) \quad \mathcal{L}_X \Theta^2 = 0, \quad \mathcal{L}_Y \Theta^2 = 0$$

and

$$(ii) \quad \mathcal{L}_X Y = 0, \quad \tilde{K}^V = 0.$$

PROOF. Since $[X, Y]^C = 0$ implies that $[X, Y] = 0$ or $\mathcal{L}_X Y = 0$. Therefore from theorem (3.2), the result follows.

Theorem (3.4). *The processes of computing the Nijenhuis tensor of Θ^2 and taking the complete lift are commutative.*

PROOF. Since $\Theta^4 - \Theta^2 = 0$. Hence the result is obvious.

4. The horizontal lift of a $\Theta(4, -2)$ -structure.

Theorem (4.1). Let $\Theta \in \mathcal{F}_1^1(M)$ be a $\Theta(4, -2)$ -structure on M . Then the horizontal lift Θ^H of Θ is also a $\Theta(4, -2)$ -structure on $cT(M)$.

PROOF. For every Θ and Ψ such that $\Theta, \Psi \in \mathcal{F}_1^1(M)$, we have [3]

$$(4.1) \quad \Theta^H \psi^H + \psi^H \Theta^H = (\Theta \psi + \psi \Theta)^H.$$

Now replacing Ψ by Θ in (4.1), we obtain

$$(4.2) \quad (\Theta^H)^2 = (\Theta^2)^H.$$

Also replacing Ψ by Θ^2 in (4.1), we get

$$\Theta^H (\Theta^2)^H + (\Theta^2)^H \Theta^H = (2\Theta^3)^H,$$

which in view of (4.2) yields

$$(4.3) \quad (\Theta^H)^3 = (\Theta^3)^H.$$

Further replacing Ψ by Θ^3 in (4.1), we obtain

$$\Theta^H (\Theta^3)^H + (\Theta^3)^H \Theta^H = (2\Theta^4)^H,$$

which in consequence of (4.3) yields

$$(4.4) \quad (\Theta^H)^4 = (\Theta^4)^H.$$

Since Θ is a $\Theta(4, -2)$ -structure on M . Therefore $\Theta^4 - \Theta^2 = 0$. Hence from (4.2) and (4.4), we have

$$(\Theta^H)^4 = (\Theta^4)^H = (\Theta^2)^H = (\Theta^H)^2,$$

or

$$(\Theta^H)^4 - (\Theta^H)^2 = 0.$$

Thus Θ^H is a $\Theta(4, -2)$ -structure on $cT(M)$.

Remark: The equation (4.4) can be generalised as follows:

$$(\Theta^H)^K = (\Theta^K)^H,$$

where K is some positive integer.

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