

On polynomials with integer coefficients and given discriminant, IV

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1. Introduction

Let us call the polynomials $f, f^* \in \mathbb{Z}[x]$ *equivalent* if $f^*(x) = f(x+a)$ for some $a \in \mathbb{Z}$. In [11] we proved that if $D \neq 0$ is a given integer then there are only finitely many pairwise non-equivalent monic polynomials $f \in \mathbb{Z}[x]$ with discriminant $D(f) = D$ and such a system of polynomials can be effectively determined. In [12] we showed that for any monic polynomial $f \in \mathbb{Z}[x]$ of degree $k \geq 2$ with non-zero discriminant D $k = \deg(f) \leq 2 + 2(\log 3)^{-1} \log |D|$ holds and there is a polynomial f^* equivalent to f such that the maximum $\|f^*\|$ of the absolute values of the coefficients of f^* satisfies

$$(1) \quad \|f^*\| < \exp \exp \{4(\log |D|)^{13}\}.$$

In [13] this last statement was proved, as a special case of a more general result, with the estimate

$$(2) \quad \|f^*\| < \exp \{c_1 |D|^\kappa\},$$

where $\kappa > 9(k-1)(k-2)/2$ and c_1 denotes an effectively computable positive constant depending only on k and κ .

In our papers [10], [11], [12], [13] and [14] various applications of these results were given to reducibility of polynomials of the form $g(f(x))$, to polynomials of given discriminant, to algebraic numbers of given discriminant and to Diophantine equations.

In Section 2 of the present paper we generalize some theorems of [11], [12] and [13] to polynomials with algebraic integer coefficients and given discriminant. In Section 3 some applications of these results are presented to algebraic integers with given relative discriminant over a fixed algebraic number field.

p -adic generalizations and further applications are given in Part V [15] and in a joint paper of Z. Z. PAPP and the author [17].

2. Polynomials with algebraic integer coefficients and given discriminant

Throughout this paper L denotes an algebraic number field of degree $n \geq 1$ with ring of integers Z_L . Let D_L be the absolute value of the discriminant of L . If $f \in Z_L[x]$ and $f^*(x) = f(x+a)$ for some $a \in Z_L$, then for their discriminants $D(f) = D(f^*)$ holds. Such polynomials $f, f^* \in Z_L[x]$ will be called Z_L -*equivalent*.

As usual, $|\overline{\alpha}|$ will denote the maximum absolute value of the conjugates of an algebraic number α and $|\overline{F}|$ will signify the maximum absolute value of the conjugates of the coefficients of a polynomial $F(x)$ with algebraic coefficients.

The following theorem contains Theorem 2 of [12], Corollary 1.6 of [13] and Corollary 1.3 of [14]. In the special case $L=Q$ (3) is obviously sharper than (2).

Theorem 1. *Let L be as above, and let δ be a non-zero integer in L with $|N_{L/Q}(\delta)| \leq d$. If $f \in Z_L[x]$ is a monic polynomial of degree $k \geq 3$ with discriminant δ , then it is Z_L -equivalent to a polynomial f^* for which*

$$(3) \quad |\overline{f^*}| < |\overline{\delta}|^{\frac{1}{k-1}} \exp \left\{ (5nk^3)^{30nk^3} ((dD_L^k)^{3/2} (\log dD_L)^{nk})^{3(k-1)(k-2)} \right\}$$

holds.

Theorem 1 is still true for $k=2$ and, by Theorem 2, in this case we have

$$(3') \quad |\overline{f^*}| < (|\overline{\delta}|^{1/2} + (2D_L^{1/2})^{n^2} + 1)^2$$

instead of (3). In what follows, all the consequences will be stated only for $k \geq 3$, however in view of (3') they remain valid even for $k=2$ with other estimates.

We note that in Theorem 1 and in its corollaries one may choose $d = |\overline{\delta}|^n$.

Corollary 1.1. *Let L be as in Theorem 1. Suppose that we are given a natural number $k \geq 2$ and a non-zero $\delta \in Z_L$. Then there are only finitely many pairwise Z_L non-equivalent monic polynomials $f \in Z_L[x]$ with $\deg(f) = k$ and $D(f) = \delta$ and such a system of polynomials can be effectively determined.*

This corollary is a generalization of certain earlier results obtained in [11], [12] and [13].

Denote by $f^{(j)}(x)$ the j -th derivative of a polynomial $f(x)$. The next corollary generalizes Corollary 1.7 of [13].

Corollary 1.2. *Let L be as in Theorem 1. If $f \in Z_L[x]$ is a monic polynomial of degree $k \geq 3$ with discriminant $D(f) = \delta \neq 0$ and $|N_{L/Q}(\delta)| = d$, then*

$$(4) \quad \begin{aligned} |\overline{D(f^{(j)})}| &< |\overline{\delta}|^{\frac{2(k-j-1)}{k-1}} \exp \left\{ c_2 ((dD_L^k)^{3/2} (\log dD_L)^{nk})^{3(k-1)(k-2)} \right\} < \\ &< \exp \left\{ 2c_2 ((|\overline{\delta}|^n D_L^k)^{3/2} (\log (|\overline{\delta}|^n D_L))^{nk})^{3(k-1)(k-2)} \right\} \end{aligned}$$

for any $0 \leq j \leq k-2$, where $c_2 = 2k(5nk^3)^{30nk^3}$.

Corollary 1.3. *Let $f(x) = x^k + a_1 x^{k-1} + \dots + a_k \in Z_L[x]$ be a polynomial of degree $k \geq 3$ with discriminant $\delta \neq 0$ and let $|N_{L/Q}(\delta)| \leq d$. If for some i , $1 \leq i \leq k$, $|\overline{a_i}| \leq A_i |\overline{f}|^\tau$ with $A_i \geq 0$ and $0 \leq \tau < i/k$, then*

$$(5) \quad |\overline{a_j}| < \binom{k+1}{j} \left\{ (kT)^{\min(i-k\tau, 1)} + 2^\tau A_i^{\min(\frac{1}{i-k\tau}, 1)} \right\}^{\max(\frac{j}{i-k\tau}, j)}$$

for each j , $1 \leq j \leq k$, where T denotes the expression occurring on the right side of inequality (3).

Corollary 1.3 generalizes some results of Parts II and III (see [12] and [13]) obtained in the special case $L=Q$.

By using the argument employed at the end of the proof of Theorem 1 of [13] we could easily extend Theorem 1 (and its corollaries) to polynomials with discriminant $\delta=0$ to get a generalization of Theorem 1 of [13]. We now generalize Theorem 1 of the present paper in another direction which also includes the case $\delta=0$.

Denote by $P_f(x) \in Z_L[x]$ the squarefree monic polynomial divisor of maximal degree¹⁾ of a monic polynomial $f \in Z_L[x]$. If $D(f) \neq 0$ we have obviously $P_f = f$. For linear P_f let $D(P_f) = 1$.

Theorem 1 is an immediate consequence of the following result.

Theorem 2. *Let L be defined as in Theorem 1, and let δ be a non-zero integer in L with $|N_{L/Q}(\delta)| \leq d$ ($d \geq 2$). If $f \in Z_L[x]$ is a monic polynomial of degree $k \geq 2$ with $\deg(P_f) = l$ and $D(P_f) = \delta$, then f is Z_L -equivalent to a polynomial f^* satisfying*

$$(6) \quad |\overline{f^*}| < |\overline{\delta}|^{\frac{k}{l(l-1)}} \exp \left\{ (5nl^3)^{30nl^3} \cdot (k/l) ((dD_L^l)^{3/2} (\log dD_L)^{nl})^{3(l-1)(l-2)} \right\}$$

for $l \geq 3$ and

$$(6') \quad |\overline{f^*}| < (|\overline{\delta}|^{1/2} + (2D_L^{1/2})^{n^2} + 1)^k$$

for $l < 3$.

It is clear that, by applying Theorem 2 in place of Theorem 1, Corollaries 1.1, 1.2 and 1.3 can be extended to polynomials with discriminant $\delta=0$ too.

3. Applications to algebraic integers with given relative discriminant

Suppose again that L is an algebraic number field of degree $n \geq 1$ and let D_L denote the absolute value of its discriminant. As usual, denote by $D_{L(\alpha)/L}(\alpha)$ and $N_{L(\alpha)/L}(\alpha)$, or, more briefly, by $D(\alpha)$ and $N(\alpha)$ the discriminant and the norm of an algebraic number α relative to the extension $L(\alpha)/L$. If α is an algebraic integer with minimal polynomial f over L , then α and f have the same degree, say k , over L , $D(\alpha) = D(f)$, $N(\alpha) = (-1)^k f(0)$, $|\overline{f}| \leq (2|\overline{\alpha}|)^k$ and $|\overline{\alpha}| \leq k|\overline{f}|$.

We shall say that the algebraic integers α and α^* are Z_L -equivalent if $\alpha - \alpha^* \in Z_L$. In this case their minimal polynomials over L are also Z_L -equivalent.

Theorems 3A and 4 are immediate consequences of Theorem 1 and Corollary 1.3 respectively.

Theorem 3A. *Let L be as above. If α is an algebraic integer with degree $k \geq 3$ and discriminant δ over L and $|N_{L/Q}(\delta)| \leq d$, then there exists an α^* Z_L -equivalent to α such that*

$$(7) \quad |\overline{\alpha^*}| < |\overline{\delta}|^{\frac{1}{k(k-1)}} \exp \left\{ (5nk^3)^{30nk^3} ((dD_L^k)^{3/2} (\log dD_L)^{nk})^{3(k-1)(k-2)} \right\}.$$

In fact Theorem 1 yields a slightly weaker estimate for $|\overline{\alpha^*}|$ in terms of $|\overline{\delta}|$. However, if we apply Theorem 2 to the minimal polynomial f of α over L , (25) implies (7).

¹⁾ In other words P_f is the monic polynomial divisor of maximal degree of f over Z_L such that $D(P_f) \neq 0$.

Theorem 3A generalizes and improves Theorem 3 of [13]*, obtained in the special case $L=Q$.

If we take into consideration even the discriminant of the number field generated by α over L , we get a sharper estimate for $|\alpha^*|$ in terms of d .

Theorem 3B. *Let L be as in Theorem 3A, and let K be an extension of degree $k \geq 3$ of L with discriminant $D_K = D_{K/Q}$. If α is an integer in K with $D_{K/L}(\alpha) = \delta \neq 0$ and $|N_{L/Q}(\delta)| \leq d$, then there exists an $\alpha^* \in K$, Z_L -equivalent to α , for which*

$$(8) \quad |\alpha^*| < |\delta|^{\frac{1}{k(k-1)}} \exp \left\{ (5c_3)^{30(c_3+2)} (|D_K|(\log |D_K|)^{nk})^{3(k-1)(k-2)} (|D_K|^{3(k-1)(k-2)/2} + \log d) \right\}$$

holds, where $c_3 = nk(k-1)(k-2)$.

Denoting by f the minimal polynomial of α over L , (8) immediately follows from (23). In particular if K/L is normal, it is easy to obtain

$$(8') \quad |\alpha^*| < |\delta|^{\frac{1}{k(k-1)}} \exp \left\{ (5nk)^{30(nk+2)} |D_K|(\log |D_K|)^{3nk-1} \cdot (|D_K|^{1/2} + \log d) \right\}$$

from the proof of Theorem 2.

An easy corollary of Theorem 3A and (3') is the following

Corollary 3.1. *Let L be as in Theorem 3A. Suppose that we are given a natural number $k \geq 2$ and a non-zero integer δ in L . Then there are only finitely many pairwise Z_L non-equivalent algebraic integers with degree k and discriminant δ over L and such a system of algebraic integers can be effectively determined.*

In the special case $L=Q$ this was earlier proved in [11] and, in an ineffective form, in [6].

We note that Corollary 3.1 can be deduced, in an ineffective form, from an ineffective theorem of B. J. BIRCH and J. R. MERRIMAN [6] on binary forms with given degree and given discriminant, combining it with Siegel's theorem concerning the number of solutions of the generalized Thue equation in integers of an algebraic number field [25]. (For an effective version of Siegel's theorem see A. BAKER [4] and A. BAKER and J. COATES [5]).

Denote by $D_{K/L}$ the relative discriminant of K/L . As is well-known (see for example H. HASSE [18]), for every primitive integral element α of K/L the principal ideal generated by $D_{K/L}(\alpha)$ can be written as

$$(9) \quad (D_{K/L}(\alpha)) = \mathfrak{I}^2(\alpha) D_{K/L}$$

with an integral ideal $\mathfrak{I}(\alpha)$ in L . $\mathfrak{I}(\alpha)$ is called the index of α with respect to K/L or the unessential divisor of the discriminant $D_{K/L}(\alpha)$. Evidently $\alpha\varepsilon$ has the same index for any unit ε of L .

The next corollary which contains Corollary 3.2 of [13] as a special case will be deduced from Theorem 3B.

Corollary 3.2. *Let L be as in Theorem 3A, $\mathfrak{I}$ a non-zero integral ideal in L with norm $\leq M$ and K an extension of degree $k \geq 3$ of L with discriminant $D_K = D_{K/Q}$.*

*) Added in proof. Very recently L.A. TRELINA (Mat. Zametki 21 (1977), 289—296) has obtained a p-adic analogue of this theorem of [13].

Suppose that there is an integer α in K with index $\mathfrak{I}$. Then α is Z_L -equivalent to an algebraic integer of the form $\alpha^* \varepsilon$, where ε is a unit in L and

$$|\overline{\alpha^*}| < \exp \{ (5nk^3)^{30nk^3} (|D_K| (\log |D_K|)^{nk})^{3(k-1)(k-2)} (|D_K|^{3(k-1)(k-2)/2} + \log M) \}.$$

An important consequence of Corollary 3.2 is that up to the obvious multiplications by units of L there are only finitely many pairwise Z_L non-equivalent integers in K with a given index $\mathfrak{I} \neq 0$ and such a system of integers can be effectively determined.

By a theorem of E. ARTIN [2] the above relative extension K/L has a relative integral basis if and only if the index of a primitive integral element α of K with respect to K/L is principal. Consequently, if $D_{K/L}$ is principal and for example the class number of L/Q is odd, then K/L has a relative integral basis (for further results and references see e.g. W. NARKIEWICZ [19]). Moreover, as is well-known, numerous special relative extensions K/L have integral bases of the form $1, \alpha, \alpha^2, \dots, \alpha^{k-1}$ with a suitable $\alpha \in Z_K$ when $Z_K = Z_L[\alpha]$. For earlier results the reader may consult [19]. Recently J. J. PAYAN [20], [21], M. N. GRAS [8], [9], G. ARCHINARD [1] and P. A. B. PLEASANTS [22] have obtained results connected with the existence of such an integral basis.*) The relative extensions having this property are generally called *monogenic*.

Corollary 3.2 yields the following general result on monogenic extensions.

Corollary 3.3. Let L be an algebraic number field of degree $n \geq 1$, and let K be an extension of degree $k \geq 3$ of L with discriminant $D_K = D_{K/Q}$. Suppose that $Z_K = Z_L[\alpha]$ for some $\alpha \in Z_K$. Then α is Z_L -equivalent to an algebraic integer of the form $\alpha^* \varepsilon$, where ε is a unit in L and

$$|\overline{\alpha^*}| < \exp \{ (5nk^3)^{30nk^3} (|D_K|^{3/2} (\log |D_K|)^{nk})^{3(k-1)(k-2)} \}.$$

This provides a general and effective algorithm for deciding whether a relative extension K/L is monogenic or not and for determining all $\alpha \in Z_K$ for which $Z_K = Z_L[\alpha]$.

The special case $L=Q$ is of particular interest. Corollary 3.3 generalizes and improves our earlier result obtained in the case $L=Q$ ([13], Corollary 3.3).

Corollary 1.3 implies the following

Theorem 4. Let L be as in Theorem 3A, and let α be an algebraic integer of relative degree $k \geq 3$ with norm μ and discriminant δ over L such that $|N_{L/Q}(\delta)| \leq d$. Denote by $f(x)$ the minimal polynomial of α over L . Let $N \geq 1$ and $0 \leq \tau < 1$ be real numbers satisfying

$$|\mu| \leq N |\overline{f}|^\tau.$$

Then

$$(10) \quad |\overline{\alpha}| \leq k |\overline{f}| < k(k+1) \{ (kT)^{\min((1-\tau)k, 1)} + 2^\tau N^{\min(\frac{1}{(1-\tau)k}, 1)} \}^{\max(\frac{1}{1-\tau}, k)},$$

where T denotes the expression occurring on the right side of (3).

Consequently, for any given algebraic number field L there are only finitely many algebraic integers with a given degree, a given norm and a given discriminant

*) Added in proof. Recently B. KNIGHT has obtained such a result in the case $L=Q$, $k=4$ (see W. M. SCHMIDT, Proc. Internat. Congress Math., Vancouver, 1974. Vol. I, pp. 177—185.)

over L and these integers can be effectively determined. In the special case $L=Q$ this was proved in [11] (see also [12] and [13]).

The next theorem provides an explicit upper bound for $|\bar{\alpha}|$ depending only on $|N_{L/Q}(\mu)|$ instead of $|\mu|$.

Theorem 5. *Let L be as in Theorem 3A, and let α be an algebraic integer with degree $k \geq 3$, with norm μ and discriminant δ over L . Suppose that $|N_{L/Q}(\delta)| \leq d$ and $|N_{L/Q}(\mu)| \leq N'$. Then we have*

$$(11) \quad |\bar{\alpha}| < \exp \left\{ (5nk^3)^{30nk^2(k+1)} ((dD_L^k)^{2/3} (\log dD_L)^{nk})^{3(k-1)} [\log(|\bar{\delta}|N') + \right. \\ \left. + ((dD_L^k)^{3/2} (\log dD_L)^{nk})^{3(k-1)(k-2)}] \right\}.$$

Choosing $d = |\bar{\delta}|^n$ in Theorem 5, we immediately get the following

Corollary. *Let L be defined as in Theorem 3A. If ε is an algebraic unit with degree $k \geq 3$ and discriminant $D(\varepsilon)$ over L , then*

$$|\bar{\varepsilon}| < \exp \{ c_4 |\bar{D}(\varepsilon)|^{n(k-1)(9k/2-7)} (\log |\bar{D}(\varepsilon)|)^{3nk^3} \}$$

and, provided that $|\bar{D}(\varepsilon)| > \exp \{9k\}$,

$$|\bar{D}(\varepsilon)| > c_5 (\log |\bar{\varepsilon}|)^{(n(k-1)(5k-7))^{-1}}$$

hold with effectively computable positive constants c_4, c_5 depending only on k, n and D_L .

By virtue of (11) it is easy to compute explicit values for c_4 and c_5 .

We remark that in the special case $L=Q$ Theorem 4 gives a slightly sharper estimate for $|\bar{\varepsilon}|$. (10) implies

$$|\bar{\varepsilon}| < \exp \{ 2k(5k^3)^{30k^3} (|D(\varepsilon)|^{3/2} (\log |D(\varepsilon)|)^k)^{3(k-1)(k-2)} \}$$

for any algebraic unit ε of degree $k \geq 3$ and discriminant $D(\varepsilon)$ over Q . This sharpens the estimates of Corollary 2.2 of [13] (see also the Corollary to Theorem 2 of [14]).

4. A preliminary result

Let M be an algebraic number field of degree m with s real and $2t$ complex conjugate fields. Write $r=s+t-1$. Let D_M denote the absolute value of the discriminant of M .

The proof of Theorem 2 depends on the following result which is proved in our paper [16].

Proposition. *Let M be as above, and let $\gamma_1, \gamma_2, \gamma_3$ be non-zero integers in M with $\max_i (|\gamma_i|) \leq G$. If Θ_1, Θ_2 and Θ_3 are non-zero integers of M satisfying*

$$\gamma_1 \Theta_1 + \gamma_2 \Theta_2 + \gamma_3 \Theta_3 = 0 \quad \text{and} \quad \max_i \{|N_{M/Q}(\Theta_i)|\} \leq N,$$

then there exists a unit ε in M such that

$$\max_i (|\overline{\Theta_i \varepsilon}|) < \exp \{c_6 D_M (\log 2D_M)^{3m-1} (D_M^{1/2} + \log(GN))\},$$

where

$$c_6 = \frac{8^{2m}}{m^{m-3}} [16m(r+3)]^{15(r+2)}.$$

This proposition sharpens and generalizes Lemma 4 of [12] and Lemma 3 of [13]. Its proof is based on a recent explicit inequality of A. J. VAN DER POORTEN and J. H. LOXTON [23], [24] on linear forms in the logarithms of algebraic numbers.

5. Proofs

PROOF OF THEOREM 2. Write

$$f(x) = (x - \alpha_1) \dots (x - \alpha_k).$$

Suppose, for convenience, that the roots of P_f are $\alpha_1, \dots, \alpha_l$. The case $l=1$ being trivial, we assume that $l \geq 2$. Then we have

$$(12) \quad D(P_f) = \prod_{1 \leq i < j \leq l} (\alpha_j - \alpha_i)^2 = \delta.$$

First suppose that $l \geq 3$. Choose any three from among $\alpha_1, \dots, \alpha_l$. We may assume without loss of generality that these are $\alpha_1, \alpha_2, \alpha_3$. Putting $M = L(\alpha_1, \alpha_2, \alpha_3)$ and $m = [M: Q]$, from (12) we get

$$(13) \quad |N_{M/Q}(\alpha_j - \alpha_i)| \leq |N_{M/Q}(\delta)|^{1/2} \leq d^{m/2n}$$

for any $1 \leq i < j \leq 3$. Further we have

$$(14) \quad (\alpha_2 - \alpha_1) + (\alpha_3 - \alpha_2) + (\alpha_1 - \alpha_3) = 0.$$

Let D_M denote the absolute value of the discriminant of M and r the free rank of the group of units of M . Apply now the Proposition of Section 4 to (14) and (13). By this Proposition there is a unit ε and integers δ_{ji} in M such that

$$(15) \quad \alpha_j - \alpha_i = \varepsilon \delta_{ji}$$

for any $1 \leq i < j \leq 3$ and

$$(16) \quad \max_{1 \leq i < j \leq 3} (|\overline{\delta_{ji}}|) < \exp \{c_7 D_M (\log 2D_M)^{3m-1} (D_M^{1/2} + \log d)\} = T_1,$$

where $c_7 = \frac{8^{2m}}{2nm^{m-4}} [16m(r+3)]^{15(r+2)}$.

Let $M_i = L(\alpha_i)$ and $m_i = [M_i: Q]$ for $i=1, \dots, l$. Denote by D_{M_i} the absolute value of the discriminant of M_i . Put $D = \max_i D_{M_i}$ and let $D=2$ if $L(\alpha_1, \dots, \alpha_l) = Q$. Then

$$(17) \quad D_M |D_{M_1}^{m/m_1} D_{M_2}^{m/m_2} D_{M_3}^{m/m_3}| \leq D^{3(l-1)(l-2)}$$

holds (see H. M. STARK [27]). (16) and (17) imply

$$T_1 < \exp \{c_8(D(\log D)^{nl})^{3(l-1)(l-2)}(D^{3(l-1)(l-2)/2} + \log d)\} = T_2,$$

where $c_8 = 2c_9^{4-c_9}(12(l-1)(l-2))^{3c_9-1}(4c_9)^{30(c_9+2)}$ and $c_9 = nl(l-1)(l-2)$.

We note that if P_f is irreducible over Z_L and the extensions M_i/L are normal then we may take $T_2 = T_1$ replacing D_M in T_1 with D , $r+1$ with m and m with nl .

If $l > 3$, we may repeat the above argument first for $\alpha_2 - \alpha_1$, $\alpha_j - \alpha_2$, $\alpha_j - \alpha_1$ with $j=4, \dots, l$, then for $\alpha_j - \alpha_i$, $\alpha_i - \alpha_1$, $\alpha_j - \alpha_1$ with $3 \leq i < j$. Then we get in the same way as in [11] and [12] that

$$\alpha_j - \alpha_i = \varepsilon \varrho_{ji}$$

for any $1 \leq i < j \leq l$, where

$$\max_{j,i} (|\overline{\varrho_{ji}}|) < T_2^{2c_9+1} \quad \text{and} \quad \max_{j,i} (|\overline{\varrho_{ji}^{-1}}|) < T_2^{3c_9-1}.$$

Thus it follows from (12) that

$$\varepsilon^{l(l-1)} = \delta \prod_{1 \leq i < j \leq l} \varrho_{ji}^{-2},$$

whence

$$|\overline{\varepsilon}| < |\overline{\delta}|^{\frac{1}{l(l-1)}} \cdot T_2^{3c_9-1}.$$

This implies

$$(18) \quad |\overline{\alpha_j - \alpha_i}| < |\overline{\delta}|^{\frac{1}{l(l-1)}} \cdot T_2^{5c_9} = T_3$$

for any $1 \leq i < j \leq l$.

If $l=2$, (18) immediately follows from (12) with $T_2=1$ instead of the above T_2 .

Write now $\alpha_1 + \dots + \alpha_l = a_1$, where obviously $a_1 \in Z_L$. By (18) we have²⁾

$$(19) \quad l\alpha_i = a_1 + \beta_i$$

for $i=1, \dots, l$, β_i being integers such that

$$(20) \quad \max_i (|\overline{\beta_i}|) < (l-1)T_3.$$

There is an integral basis $1, \omega_2, \dots, \omega_n$ for L with the property

$$\max_s (|\overline{\omega_s}|) < n2^{n(n-1)} D_L^{(n^2-1)/2}$$

(see [12]). Represent a_1 in such a basis. We can easily see that there is an $a_2 \in Z_L$ congruent to $a_1 \pmod{l}$ for which

$$(21) \quad |\overline{a_2}| < ln^2 2^{n(n-1)} D_L^{(n^2-1)/2}.$$

²⁾ The author is indebted to Z. Z. PAPP for his remark that in the proof of Theorem 2 of Part II (see [12], p. 139) one can use a deduction of the type (18) $\Rightarrow$ (19) in place of employing an integral basis. This observation enables us to slightly simplify the above deduction (18) $\Rightarrow$ (22), (23) too.

Write $a_1 = la + a_2$. Then $a \in Z_L$. Since each $\alpha_j, j > l$, is equal to one of the $\alpha_i, 1 \leq i \leq l$, by (19), (20), (21) and $D_L | D$ every α_i can be written in the form

$$(22) \quad \alpha_i = a + \gamma_i, \quad i = 1, \dots, k,$$

where $\gamma_1, \dots, \gamma_k$ are algebraic integers satisfying

$$(23) \quad \max_{1 \leq i \leq k} (|\overline{\gamma_i}|) < |\overline{\delta}|^{\frac{1}{l(l-1)}} \exp \left\{ (5c_9)^{30(c_9+2)} (D(\log D)^{nl})^{3(l-1)(l-2)} (D^{3(l-1)(l-2)/2} + \log d) \right\}$$

if $l \geq 3$ and

$$(23') \quad \max_{1 \leq i \leq k} (|\overline{\gamma_i}|) < |\overline{\delta}|^{1/2} + (2D_L^{1/2})^{n^2}$$

if $l = 2$.

In the case $l = 1$ we may choose $\gamma_i = 0$ in (22) for any i .

We are going to derive an upper bound for $\max (|\overline{\gamma_i}|)$ not depending on D . Since D does not occur in (23'), it suffices to consider the case $l \geq 3$. We recall that $D = D_{M_i}$ for some $M_i = L(\alpha_i)$, where $1 \leq i \leq l$. The relative discriminant $D_{M_i/L}$ divides $D(P_f)$, consequently

$$(24) \quad D = D_{M_i} = N_{L/Q}(D_{M_i/L}) D_L^{[M_i:L]} \leq d D_L^l.$$

Finally, from (23) and (24) we get

$$(25) \quad \max_{1 \leq i \leq k} (|\overline{\gamma_i}|) < |\overline{\delta}|^{\frac{1}{l(l-1)}} \exp \left\{ (c_{10}/2l) ((d D_L^l)^{3/2} (\log d D_L)^{nl})^{3(l-1)(l-2)} \right\}$$

with $c_{10} = (5nl^3)^{30nl^3}$.

In view of (22) the polynomial

$$f^*(x) = (x - \gamma_1) \dots (x - \gamma_k) \in Z_L[x]$$

is Z_L -equivalent to f and

$$|\overline{f^*}| \leq (1 + \max_{1 \leq i \leq k} (|\overline{\gamma_i}|))^k.$$

This together with (25) and (23') imply (6) and (6').

PROOF OF COROLLARY 1.2. By Theorem 1 there is a polynomial $f^* Z_L$ -equivalent to f such that

$$|\overline{f^*}| < T_4,$$

where, for brevity, T_4 denotes the upper bound given in Theorem 1 for $|\overline{f^*}|$. For $1 \leq j \leq k-2$ this implies

$$|\overline{f^{*(j)}}| < k^j T_4,$$

whence

$$|\overline{D(f^{(j)})}| = |\overline{D(f^{*(j)})}| < (k^{2k} T_4)^{2(k-j-1)}$$

and (4) follows.

PROOF OF COROLLARY 1.3. By Theorem 1 there exists a polynomial $f^*(x) = x^k + b_1 x^{k-1} + \dots + b_k \in Z_L[x]$ such that $f(x) = f^*(x+a)$ for some $a \in Z_L$ and

$$(26) \quad |f^*| < T_4,$$

where T_4 , as above, denotes the expression occurring on the right side of (3). The case $a=0$ being trivial, we suppose that $a \neq 0$.

We have

$$|f| \leq |a|^k + 2^k T_4 |a|^{k-1}.$$

In case $|a| < 2^k T_4$ (33) immediately follows, hence we suppose $|a| \geq 2^k T_4$, when

$$(27) \quad |f| \leq 2 |a|^k.$$

On the other hand we have

$$(28) \quad a_i = \frac{f^{*(k-i)}(a)}{(k-i)!} = \binom{k}{k-i} a^i + \binom{k-1}{k-i} b_1 a^{i-1} + \dots + b_i.$$

Denote by $a^{(i)}$ such a conjugate of a over Q for which $|a^{(i)}| = |a|$ holds. By taking the corresponding conjugates on both sides of (28) we get

$$(29) \quad a_i^{(i)} = \binom{k}{k-i} (a^{(i)})^i + \binom{k-1}{k-i} b_1^{(i)} (a^{(i)})^{i-1} + \dots + b_i^{(i)}.$$

Since $|a^{(i)}| \geq 1$, it follows from (26), (27) and (29) that

$$(30) \quad \begin{aligned} |a| &= \binom{k}{k-i}^{-1} \left| \binom{k-1}{k-i} b_1^{(i)} + \binom{k-2}{k-i} b_2^{(i)} (a^{(i)})^{-1} + \dots + b_i^{(i)} (a^{(i)})^{-i+1} - a_i^{(i)} (a^{(i)})^{-i+1} \right| \\ &\leq \frac{i}{k-i+1} T_4 + A_i |f|^\tau / |a|^{i-1} \leq \frac{i}{k-i+1} T_4 + 2^\tau A_i |a|^{k\tau - (i-1)}. \end{aligned}$$

First suppose that $k\tau - (i-1) \geq 0$, that is $\frac{1}{i-k\tau} \geq 1$. Applying Lemma 5 of [13] to the linear polynomial $x - \frac{i}{k-i+1} T_4$, we get

$$(31) \quad |a| \leq \left[\left(\frac{i}{k-i+1} T_4 \right)^{i-k\tau} + 2^\tau A_i \right]^{\frac{1}{i-k\tau}}.$$

Let now $k\tau - (i-1) < 0$, that is $\frac{1}{i-k\tau} < 1$. Since $|a| \leq A_i^{\frac{1}{i-k\tau}}$ immediately implies (32), hence we assume that $|a| > A_i^{\frac{1}{i-k\tau}}$. This yields

$$A_i |a|^{k\tau - (i-1)} < A_i^{\frac{1}{i-k\tau}},$$

whence, by (30),

$$(32) \quad |\overline{a}| \leq \left[\frac{i}{k-i+1} T_4 + 2^{\tau} A_i^{\frac{1}{i-k\tau}} \right].$$

From (31) and (32) we get

$$(33) \quad |\overline{a}| \leq [(kT_4)^{\min(i-k\tau, 1)} + 2^{\tau} A_i^{\min(\frac{1}{i-k\tau}, 1)}]^{\max(\frac{1}{i-k\tau}, 1)} = T_5.$$

Finally we obtain

$$|\overline{a_j}| = \left| \frac{f^{*(k-j)}(a)}{(k-j)!} \right| \leq \binom{k+1}{j} T_5^j$$

which was to be proved.

PROOF OF COROLLARY 3.2. By (9) the discriminant $D_{K/L}(\alpha)$ satisfies

$$(D_{K/L}(\alpha)) = \mathfrak{S}^2 D_{K/L},$$

whence

$$(34) \quad |N_{L/Q}(D_{K/L}(\alpha))| = N_{L/Q}(\mathfrak{S}^2) N_{L/Q}(D_{K/L}) \leq M^2 |D_K|.$$

Let r denote the free rank of the group of units of L . If $r > 0$, by a theorem of C. L. SIEGEL [26] there exist independent units $\eta_1, \dots, \eta_r$ in L such that

$$|\log |\eta_l|| < 3n \left(\frac{5 \log D_L}{2n-2} \right)^{n-1} D_L^{1/2} = T_6$$

for $l=1, \dots, r$. Let U denote the multiplicative group generated by $\eta_1, \dots, \eta_r$. A well-known argument shows (see e.g. Z. I. BOREVICH and I. R. SHAFAREVICH [7] or A. BAKER [3]) that there is a unit $\eta \in U$ such that

$$(35) \quad |\overline{D_{K/L}(\alpha) \cdot \eta}| < [M^2 |D_K|]^{1/n} \exp \{n^2 T_6\}.$$

Further η can be written in the form $\eta = \eta' \varepsilon^{-k(k-1)}$ with $\eta', \varepsilon \in U$ for which

$$|\overline{\eta'}| < \exp \{rk(k-1)T_6\}.$$

Writing $\delta = D_{K/L}(\varepsilon^{-1}\alpha)$, we obtain from (35)

$$|\overline{\delta}| < [M^2 |D_K|]^{1/n} \exp \{n^2 k(k-1)T_6\}.$$

In view of (34) this is still true for $r=0$ with $\varepsilon=1$ and $T_6=0$. Applying now Theorem 3B to $\varepsilon^{-1}\alpha$, we get an α^* Z_L -equivalent to $\varepsilon^{-1}\alpha$ for which

$$|\overline{\alpha^*}| < \exp \left\{ (5nk^3)^{30nk^3} (|D_K| (\log |D_K|)^{nk})^{3(k-1)(k-2)} (|D_K|^{3(k-1)(k-2)/2} + \log M) \right\}.$$

This proves Corollary 3.2.

PROOF OF THEOREM 5. By Theorem 3A α can be written in the form $\alpha = \alpha^* + a$, where $a \in Z_L$ and for $|\overline{\alpha^*}|$ (7) holds. For brevity, denote by T_7 the upper bound given in Theorem 3A for $|\overline{\alpha^*}|$.

By taking the norm of α over L we get

$$(\alpha_1^* + a) \dots (\alpha_k^* + a) = \mu,$$

where $\alpha_1^*, \dots, \alpha_k^*$ denote the conjugates of α^* over L . Suppose, for convenience, that $\alpha^* = \alpha_1^*$. Writing $\alpha_i^* + a = \mu_i$ for $i = 1, 2$, we obtain

$$(36) \quad \mu_1 - \mu_2 = \alpha_1^* - \alpha_2^*.$$

Put $M = L(\alpha_1^*, \alpha_2^*)$ and $m = [M: Q]$. Let D_M be the absolute value of the discriminant of M . Then $m \leq nk(k-1)$ and we obtain $D_M \leq (dD_L^k)^{2(k-1)}$ by the same reasoning as in (17) and (24). Further $|N_{M/Q}(\mu_i)| \leq N'^{k(k-1)}$. Apply now the Proposition of Section 4 to (36) with $\gamma_1 = \gamma_2 = \Theta_3 = 1$ and $\gamma_3 = \alpha_2^* - \alpha_1^*$. By virtue of this Proposition there is a unit ε in M such that

$$\max(|\mu_1 \varepsilon|, |\mu_2 \varepsilon|, |\varepsilon|) < \exp \{c_{11} D_M (\log 2D_M)^{3m-1} (D_M^{1/2} + \log(2T_7 N'^{k(k-1)}))\} = T_8$$

with $c_{11} = \frac{8^{2m}}{m^{m-3}} (4m)^{30(m+2)}$. Since $|\varepsilon^{-1}| < T_8^{m-1}$, hence we have

$$|\alpha| = |\mu_1| < T_8^m < \exp \left\{ (5nk^3)^{30nk^2(k+1)} ((dD_L^k)^{2/3} (\log dD_L)^{nk})^{3(k-1)} [\log(|\delta|N') + ((dD_L^k)^{3/2} (\log dD_L)^{nk})^{3(k-1)(k-2)}] \right\}.$$

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(Received January 28, 1975; in revised form February 17, 1977)