

A fixed point theorem for compact metric spaces

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In a recent paper, see [1], the following theorem was proved:

Theorem 1. *If T is a mapping of the complete metric space X into itself satisfying the condition*

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$$

for all x, y in X , where $0 \leq c < 1/2$, then T has a unique fixed point.

We will now prove the following theorem:

Theorem 2. *If T is a continuous mapping of a compact metric space X into itself satisfying the condition*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Ty) + d(y, Tx)]$$

for all distinct x, y in X , then T has a unique fixed point.

PROOF. Define a real-valued function f on X by

$$f(x) = d(x, Tx).$$

Since d and T are continuous functions it follows that f is a continuous function on X . Since X is compact it achieves its minimum value and so there exists a point z in X such that

$$f(z) = \inf \{f(x) : x \in X\}.$$

We will now suppose that $Tz \neq z$. Then by hypothesis

$$\begin{aligned} d(Tz, T^2z) &< \frac{1}{2}[d(z, T^2z) + d(Tz, Tz)] \\ &= \frac{1}{2}d(z, T^2z) \leq \frac{1}{2}[d(z, Tz) + d(Tz, T^2z)] \end{aligned}$$

and so

$$d(Tz, T^2z) < d(z, Tz)$$

or equivalently

$$f(Tz) < f(z).$$

This contradicts the definition of z and so we must have $Tz = z$. Thus z is a fixed point of T .

We will now prove that z is unique. Suppose that z' is a second fixed point with $z \neq z'$. Then

$$d(z, z') = d(Tz, Tz') < \frac{1}{2} [d(z, Tz') + d(z', Tz)] = d(z, z'),$$

giving a contradiction and so proves the uniqueness of z . This completes the proof of the theorem.

We will now show from the following *example* that the condition that X be compact is necessary.

We will let X be the set of real numbers

$$X = \{n - n^{-1} : n = 1, 2, \dots\}$$

with metric

$$d(x, y) = |x - y|$$

for all x, y in X . X is obviously complete with this metric but not compact.

Define a mapping T of X into itself by

$$T\{n - n^{-1}\} = n + 1 - (n + 1)^{-1}$$

for $n = 1, 2, \dots$. Then supposing that $m < n$, we have

$$d(T(m - m^{-1}), T(n - n^{-1})) = n - m - (n + 1)^{-1} + (m + 1)^{-1}$$

and

$$\begin{aligned} & d(m - m^{-1}, T(n - n^{-1})) + d(n - n^{-1}, T(m - m^{-1})) = \\ & = -m + m^{-1} + n + 1 - (n + 1)^{-1} + n - n^{-1} - (m + 1) + (m + 1)^{-1} = \\ & = 2[n - m - (n + 1)^{-1} + (m + 1)^{-1}] + [(n + 1)^{-1} - n^{-1} + m^{-1} - (m + 1)^{-1}]. \end{aligned}$$

Since $m < n$, it follows that

$$(n + 1)^{-1} - n^{-1} + m^{-1} - (m + 1)^{-1} > 0$$

and so

$$d(T(m - m^{-1}), T(n - n^{-1})) < \frac{1}{2} [d(m - m^{-1}, T(n - n^{-1})) + d(n - n^{-1}, T(m - m^{-1}))]$$

if $m < n$ and by the symmetry this inequality now of course holds if $m \neq n$.

T is obviously continuous and so this example satisfies all the hypotheses of the theorem with the exception of the compactness of X , but T has no fixed point.

We finally note that the condition that the inequality does not have to hold if $x = y$ is also essential, otherwise we would have

$$d(Tx, Tx) < d(x, Tx)$$

and no fixed point of T could satisfy this inequality.

Reference

- [1] B. FISHER, A fixed point theorem, *Mathematics Magazine*, **48**, (1975), 223—225.

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