

On characterizations of certain classes of semigroups

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Several important classes of semigroups or associative rings have been characterized by ideal-theoretical relations (see [2], [3], [4], [5]). These characterizations are given by equalities. In this paper three important classes of semigroups will be characterized by ideal-theoretical inequalities. These classes consist of regular and/or intraregular semigroups.

For the notions and notations used in this paper we refer to A. H. CLIFFORD and G. B. PRESTON [1].

Theorem 1. *For a semigroup S the following conditions are pairwise equivalent:*

- (A) S is regular.
- (B) $B \cap R \subseteq RB$ for every bi-ideal B and every right ideal R of S .
- (C) $B(a) \cap R(b) \subseteq R(b)B(a)$ for every couple a, b of elements in S .
- (D) $B(a) \cap R(a) \subseteq R(a)B(a)$ for every element a of S .

PROOF. (A) implies (B). Let S be a regular semigroup. Then any bi-ideal B of S can be written in the form

$$(1) \quad B = L' \cap R' = R'L',$$

where L' is a left ideal, R' is a right ideal of S (cf. [5], Theorem 29). Since one-sided ideals of S are evidently globally idempotent, we have (by making use of the Kovács—Iséki criterion [1], p. 34)

$$(2) \quad B \cap R = L' \cap R' \cap R \subseteq L' \cap RR' = RR'L' = RB$$

for every bi-ideal B and every right ideal R of S .

The implications (B) $\Rightarrow$ (C) $\Rightarrow$ (D) are trivial.

(D) implies (A). Let S be a semigroup with property (D). Then (D) implies

$$(3) \quad L(a) \cap R(a) \subseteq R(a)L(a)$$

for any element a of S . But $R(a)L(a) \subseteq L(a) \cap R(a)$ also holds, thus we get

$$(4) \quad L(a) \cap R(a) = R(a)L(a)$$

for every element a of S . This means (see [3]) that every element a of S is regular.

REMARK 1. It is easy to see that Theorem 1 remains true with quasi-ideal instead of bi-ideal.

Theorem 2. For a semigroup S the following conditions are equivalent:

- (A) S is intraregular.
- (B) $L \cap R \subseteq LR$ for every left ideal L and every right ideal R of S .
- (C) $L(a) \cap R(b) \subseteq L(a)R(b)$ for every couple a, b of elements in S .
- (D) $L(a) \cap R(a) \subseteq L(a)R(a)$ for every element a of S .

PROOF. (A) implies (B). Suppose that S is an intraregular semigroup, L is a left ideal, R is a right ideal of S , and $a \in L \cap R$. Then the product LR is a two-sided ideal of S , and $a^2 \in LR$ implies $a \in LR$ because the ideals of S are semiprime (cf. [1], p. 121). Thus (A) implies (B) indeed.

Evidently (B) $\Rightarrow$ (C) $\Rightarrow$ (D).

(D) implies (A). Let S be a semigroup with property (D). Then we have

$$(5) \quad a \in L(a) \cap R(a) \subseteq L(a)R(a) \subseteq S^1 a^2 S^1$$

for every element a of S . Hence it follows easily that S is intraregular.

For an earlier ideal-theoretic characterization of intraregular semigroups, see G. SZÁSZ [7].

Theorem 3. For a semigroup S the following conditions are equivalent:

- (A) S is regular and intraregular.
- (B) $A \cap B \subseteq AB$ for every couple A, B of bi-ideals of S .
- (C) $B \cap Q \subseteq BQ$ for every bi-ideal B and every quasi-ideal Q of S .
- (D) $P \cap Q \subseteq PQ$ for every couple P, Q of quasi-ideals of S .
- (E) $L \cap R \subseteq LR \cap RL$ for every left ideal L and every right ideal R of S .
- (F) $L(a) \cap R(b) \subseteq L(a)R(b) \cap R(b)L(a)$ for every couple a, b of elements in S .
- (G) $L(a) \cap R(a) \subseteq L(a)R(a) \cap R(a)L(a)$ for every element a of S .

PROOF. (A) implies (B). Let S be a semigroup which is both regular and intraregular. Then the bi-ideals of S are globally idempotent (see [5], Theorem 39). Hence it follows (B), because

$$(6) \quad A \cap B = (A \cap B)^2 \subseteq AB$$

holds for any two bi-ideals A, B of S .

Evidently (B) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (E) $\Rightarrow$ (F) $\Rightarrow$ (G).

(G) implies (A). Let S be a semigroup with property (G). Then (G) implies (3) resp. (4) for any element a of S . Thus S is regular. On the other hand, (G) implies condition (D) of our Theorem 2. Hence S is intraregular.

Theorem 3 is completely proved.

REMARK 2. The first author proved in [4] that a semigroup S is a semilattice of groups if and only if the condition

$$(7) \quad L \cap R = LR$$

holds for every left ideal L and every right ideal R of S . This criterion and our Theorem 2 imply that a semigroup which is a semilattice of groups is necessarily intraregular. Similarly it can be shown that also the semilattices of left [right] groups are intraregular semigroups.

For further characterizations by ideal-theoretical inequalities, see [6].

References

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