

On additive functions

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1. Let $f(n)$ and $g(n)$ denote additive arithmetical functions. Some years ago I proved that from

$$(1.1) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| = 0$$

it follows that $f(n)$ is a constant multiple of $\log n$ [1]. This was an old conjecture of P. ERDŐS [2]. Almost at the same time E. WIRSING [3] obtained a stronger result, namely that from

$$(1.2) \quad \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{x \leq n \leq (1+\gamma)x} |f(n+1) - f(n)| = 0$$

— γ being a positive constant — it follows that $f(n) = c \log n$.

The purpose of this paper is to generalize my previous result in two directions, which we state as Theorem 1 and 2.

Theorem 1. *Let $f(n)$ and $g(n)$ be additive functions and the relation*

$$(1.3) \quad \liminf_{x \rightarrow \infty} (\log x)^{-1} \sum_{n \leq x} \frac{1}{n} |g(n+1) - f(n)| = 0$$

hold. Then $f(n) = g(n) = c \log n$.

Let $\Delta^k f(n)$ denote the k 'th difference of $f(n)$, i.e. $\Delta^1 f(n) = f(n+1) - f(n)$, and generally

$$\Delta^j f(n) = \Delta^{j-1} f(n+1) - \Delta^{j-1} f(n).$$

Theorem 2. *Let $f(n)$ be an additive function and for some positive integer k the relation*

$$(1.4) \quad \liminf_{x \rightarrow \infty} (\log x)^{-1} \sum_{n \leq x} \frac{|\Delta^k f(n)|}{n} = 0$$

hold. Then $f(n)$ is a constant multiple of $\log n$.

Namely for $k=1$ Theorem 2 gives that $f(n)$ is a constant multiple of $\log n$, if

$$(1.5) \quad \liminf_{x \rightarrow \infty} (\log x)^{-1} \sum_{n \leq x} \frac{|f(n+1) - f(n)|}{n} = 0.$$

This a little stronger assertion than that was published in [1] but weaker than the cited result due to Wirsing.

First we prove Theorem 2 in the case $k=1$. After then we prove that from the condition (1.4) it follows that it holds for $k-1$ instead of k . Finally we prove Theorem 1 by showing that from (1.3) $f(n)=g(n)$ follows.

2. Proof of Theorem 2 for $k=1$

We need a lemma.

Lemma 1. *If (1.5) holds then $f(n)$ is completely additive.*

PROOF. Let

$$(2.1) \quad g_a(n) = \max_{-a \leq j \leq a} |f(n+j) - f(n)|.$$

Then from (1.5) we get easily that

$$(2.2) \quad \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} g_a(n) = 0,$$

for every fixed a .

We need to prove that $f(p^v) = vf(p)$ for all prime-power p^v . For an arbitrary N coprime to p we have

$$f(p^v) + f(N) = f(p^v N) = \{f(p^v N) - f(p^v N + p)\} + f(p) + f(p^{v-1} N + 1) = \dots$$

$$\dots = \{f(p^v N) - f(p^v N + p)\} + \sum_{j=1}^{v-2} \{f(p^{v-j} N + 1) - f(p^{v-j} j + p)\} +$$

$$+ \{f(pN + 1) - f(pN)\} + f(N) + vf(p).$$

Hence

$$|f(p^v) - vf(p)| \leq g_p(p^v N) + g_p(p^{v-1} N) + \dots + g_p(pN),$$

and so

$$|f(p^v) - vf(p)| \cdot \sum_{\substack{N \leq x \\ (N, p)=1}} 1 \leq \sum_{m \leq xp} g_p(m).$$

Since

$$\sum_{\substack{N \leq x \\ (N, p)=1}} 1 = x(1 - 1/p) + O(1),$$

by (2.2) we have $f(p^v) = vf(p)$. This completes the proof of our lemma.

Let $x_v = p^v$ ($v=0, 1, 2, \dots$) where p is an arbitrary prime. Let for $x_v \leq N < x_{v+1}$

$$(2.3) \quad N = \sum_{j=0}^v a_j(N) p^{v-j}$$

be the p -adical representation of N . Then

$$0 \leq a_j(N) \leq p-1 \quad (0 \leq j \leq v), \quad a_0(N) \equiv 1.$$

For an arbitrary N let the sequence $(N=)N_0, N_1, \dots, N_v$ be defined as follows:

$$(2.4) \quad N_k = \sum_{j=0}^{v-k} a_j(N) p^{v-j-k} \quad (k = 0, \dots, v).$$

So we have

$$(2.5) \quad N_{k-1} = pN_k + a_{v-k+1}(N) \quad (k = 1, \dots, v).$$

Since

$$\begin{aligned} f(N_0) &= \{f(N_0) - f(pN_1)\} + \{f(N_1) - f(pN_2)\} + \dots \\ &\quad + \{f(N_{v-1}) - f(pN_v)\} + f(N_v) + vf(p), \end{aligned}$$

we have

$$|f(N_0) - vf(p)| \leq g_p(N_0) + g_p(N_1) + \dots + g_p(N_{v-1}) + A_p,$$

where

$$A_p = \max_{j=1, \dots, p-1} |f(j)|.$$

For $N \in (x_v, x_{v+1})$ we get $N_k \in [x_{v-k}, x_{v-k+1})$. Furthermore, any fixed m in $[x_{v-k}, x_{v+1-k})$ occurs exactly p^k times as N_k . Therefore

$$\begin{aligned} \frac{1}{x_{v+1}} \sum_{x_v \leq N < x_{v+1}} |f(N) - vf(p)| &\leq \frac{1}{x_{v+1}} \sum_{x_v \leq N < x_{v+1}} g_p(N) + \\ &+ \frac{1}{x_v} \sum_{x_{v-1} \leq N < x_v} g_p(N) + \dots + A_p \leq \sum_{N \leq x_{v+1}} \frac{g_p(N)}{N} + A_p. \end{aligned}$$

Since $v = \left\lfloor \frac{\log N}{\log p} \right\rfloor$ for $x_v \leq N < x_{v+1}$, therefore we have

$$(2.6) \quad \frac{1}{x_{v+1}} \sum_{x_v \leq N < x_{v+1}} \left| \frac{f(N)}{\log N} - \frac{f(p)}{\log p} \right| \leq \frac{c_p}{\log x_{v+1}} \sum_{N \leq x_{v+1}} \frac{g_p(N)}{N}.$$

c_p and later c_q are suitable constants.

Similarly, when q is another prime, $y_\mu = q^\mu$, then

$$\frac{1}{y_{\mu+1}} \sum_{y_\mu \leq N < y_{\mu+1}} \left| \frac{f(N)}{\log N} - \frac{f(q)}{\log q} \right| \leq \frac{c_q}{\log y_{\mu+1}} \sum_{N \leq y_{\mu+1}} \frac{g_q(N)}{N}.$$

Let $q < p$. Then the interval (x_v, x_{v+1}) contains an integer-power of q , say y_μ . Then there is at least one subinterval I_v in $[x_v, x_{v+1})$ with length greater than cx_{v+1} (c being a positive constant) the endpoints of which are powers of p and q . Then by (2.5) and (2.6)

$$\frac{1}{x_{v+1}} \sum_{N \in I_v} \left| \frac{f(p)}{\log p} - \frac{f(q)}{\log q} \right| \leq \frac{c_1}{\log x_{v+1}} \sum_{N \leq x_{v+1}} \frac{g_p(N)}{N}.$$

Since

$$\liminf_v (\log x_{v+1})^{-1} \sum_{N \leq x_{v+1}} N^{-1} g_p(N) = 0 \quad \text{and} \quad \sum_{N \in I_v} 1 \geq cx_{v+1},$$

we have

$$\frac{f(p)}{\log p} = \frac{f(q)}{\log q}.$$

This completes the proof of our assertion.

3. Reduction step

Let H_k denote the assertion stated in (1.4). We prove that H_k implies H_{k-1} for $k \geq 2$.

Let $\Delta_2 f(n) = f(n+2) - f(n)$, and

$$(3.1) \quad \Delta_2^j f(n) = \Delta_2^{j-1} f(n+2) - \Delta_2^{j-1} f(n).$$

Lemma 2. For every n

$$(3.2) \quad \Delta_2^k f(n) = \sum_{j=0}^k \binom{k}{j} \Delta^k f(n+j),$$

and for every odd n

$$(3.3) \quad \Delta_2^k f(n) = \sum_{j=0}^k \binom{k}{j} \Delta_2^k f(2n+2j)$$

hold.

The proof of these relations is straightforward so we omit it.

From (3.2) we get — using it by $k-1$ instead of k —

$$(3.4) \quad \Delta_2^{k-1} f(n+1) - \Delta_2^{k-1} f(n) = \sum_{j=0}^{k-1} \binom{k-1}{j} \Delta^k f(n+j).$$

Furthermore from (3.2) we get the relation

$$(3.5) \quad |\Delta_2^k f(n)| \leq \sum_{j=0}^k \binom{k}{j} |\Delta^k f(n+j)|.$$

Hence

$$(3.6) \quad |\Delta_2^{k-1} f(n+2) - \Delta_2^{k-1} f(n)| \leq 2^{k+1} \sum_{j=0}^k |\Delta^k f(n+j)|.$$

From (3.3) we get for odd n — using it by $k-1$ instead of k —

$$\Delta_2^{k-1} f(n) - 2^{k-1} \Delta_2^{k-1} f(2n) = \sum_{j=0}^{k-1} \binom{k-1}{j} \{ \Delta_2^{k-1} f(2n+2j) - \Delta_2^{k-1} f(2n) \},$$

and hence

$$(3.8) \quad |\Delta_2^{k-1} f(n) - 2^{k-1} \Delta_2^{k-1} f(2n)| \leq 2^{k-1} \sum_{h=0}^{k-1} |\Delta_2^k f(2n+2h)|.$$

Let

$$x_v = 2^v \quad (v = 1, 2, \dots), \quad I_v = [x_v, x_{v+1})$$

and

$$(3.9) \quad \alpha(v) = \sum_{n \in I_v} |\Delta^{k-1} f(n+1) - \Delta^{k-1} f(n)| = \sum_{n \in I_v} |\Delta^k f(n)|.$$

Let B, C, E denote the set of even, odd, two-times-odd integers, respectively. Correspondingly let $\beta(v), \gamma(v), \varepsilon(v)$ the sum

$$\sum_n |\Delta_2^{k-1} f(n)|$$

where we take the summation over the sets $B \cap I_v, C \cap I_v, E \cap I_v$, respectively.

From (3.6) and (3.2) we get

$$\begin{aligned}\gamma(v+1) &= \sum_{\substack{2m \in I_{v+1} \\ m \in B}} |\Delta_2^{k-1} f(2m)| \leq 2 \sum_{\substack{m \in I_v \\ m \in B}} |\Delta_2^{k-1} f(2m)| + 2 \sum_{m \in I_v} |\Delta_2^k f(2m)| \leq \\ &\leq 2\varepsilon(v+1) + 2^{2k+1} \sum_{x_v+1 \leq m < x_{v+2}+k} |\Delta^k f(m)|.\end{aligned}$$

Assume now that $k < x_{v+1}$. Then

$$(3.10) \quad \gamma(v+1) \leq 2\varepsilon(v+1) + 2^{2k+1}(\alpha(v+1) + \alpha(v+2)).$$

From the inequality (3.8) we get

$$\begin{aligned}(3.11) \quad \varepsilon(v+1) &= \sum_{\substack{m \in I_v \\ m \in B}} |\Delta_2^{k-1} f(2m)| \leq \frac{1}{2^{k-1}} \sum_{\substack{m \in I_v \\ m \in B}} |\Delta_2^{k-1} f(m)| + \\ &+ k \sum_{x_v \leq m \leq x_{v+2}-2k} |\Delta_2^k f(2m)| \leq \frac{1}{2^{k-1}} \beta(v) + 2^k k(\alpha(v+1) + \alpha(v+2)).\end{aligned}$$

So by (3.10) we get

$$(3.12) \quad \gamma(v+1) \leq \frac{1}{2^{k-2}} \beta(v) + 2^{2k+2}(\alpha(v+1) + \alpha(v+2)).$$

Similarly from (3.4) we get easily that

$$(3.13) \quad |\beta(v+1) - \gamma(v+1)| \leq 2^{k+1}(\alpha(v+1) + \alpha(v+2)).$$

So we have

$$(3.14) \quad \beta(v+1) \leq \frac{1}{2^{k-2}} \beta(v) + c_1(k)(\alpha(v+1) + \alpha(v+2))$$

if $k < x_{v+1}$, where $c_1(k)$ is a constant that depends only on k .

From the condition (1.4) it follows that

$$(3.15) \quad \liminf_{\mu} \frac{1}{\mu} \sum_{v \leq \mu} \frac{\alpha(v)}{x_v} = 0.$$

Let $\varrho_v = \frac{\beta(v)}{2^v}$. From (3.15) we have

$$(3.16) \quad \varrho_{v+1} \leq \frac{1}{2^{k-1}} \varrho_v + \tau_v, \quad (x_v \geq k)$$

where

$$\tau_v = c_2(k) \left(\frac{\alpha(v+1)}{2^{v+1}} + \frac{\alpha(v+2)}{2^{v+2}} \right).$$

Let $\mu_1 < \mu_2 < \dots$ be a sequence of integers so that

$$(3.17) \quad \frac{1}{\mu_t} \sum_{v \leq \mu_t} \tau_v \rightarrow 0 \quad (t \rightarrow \infty).$$

From (3.15) we can deduce easily that

$$\lim_{t \rightarrow \infty} \frac{1}{\mu_t} \sum_{v \in \mu_t} \varrho_v = 0.$$

Observing (3.13), we have

$$(3.18) \quad \lim_{x_{\mu_t} \rightarrow \infty} (\log x_{\mu_t})^{-1} \sum_{n \in x_{\mu_t}} \frac{|\Delta_2^{k-1} f(n)|}{n} = 0.$$

From (3.2) we get

$$\begin{aligned} \Delta_2^{k-1} f(n) - 2^{k-1} \Delta^{k-1} f(n) &= \sum_{j=0}^{k-1} \binom{k-1}{j} \{\Delta^{k-1} f(n+j) - \Delta^{k-1} f(n)\} = \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} \left\{ \sum_{h=0}^{j-1} \Delta^k f(n+h) \right\}, \end{aligned}$$

and hence

$$|\Delta^{k-1} f(n)| \leq \frac{1}{2^{k-1}} |\Delta_2^{k-1} f(n)| + k \cdot \sum_{j=0}^{2k-3} |\Delta^k f(n+j)|,$$

and so

$$\sum_{n \in x} \frac{|\Delta^{k-1} f(n)|}{n} \leq \frac{1}{2^{k-1}} \sum_{n \in x} \frac{|\Delta_2^{k-1} f(n)|}{n} + 2k^2 \sum_{n \in x+2k} \frac{|\Delta^k f(n)|}{n}.$$

Let now x be choosen as $x = x_{\mu_t} - 2k$. Then from (3.17) and (3.18) we get

$$(3.19) \quad \lim_{t \rightarrow \infty} (\log x_{\mu_t})^{-1} \sum_{n \in x_{\mu_t} - 2k} \frac{|\Delta^{k-1} f(n)|}{n} = 0.$$

So we proved that the condition H_{k-1} holds.

By this, Theorem 2 has been proved.

4. Proof of Theorem 1

We need to prove only that from (1.3) the relation $f(n) = g(n)$ follows. Let $\varrho(n) = g(n+1) - f(n)$, and $H(n) = g(n) - f(n)$. We observe that

$$(4.1) \quad g(16k+12) - f(16k+10) = [g(4) - g(2) - f(2)] + \varrho(8k+5)$$

and that

$$(4.2) \quad g(16k+12) - f(16k+10) = \varrho(16k+11) - H(16k+11) + \varrho(16k+10).$$

Let $C = g(4) - g(2) - f(2)$.

From (4.1) and (4.2) we have

$$(4.3) \quad C + H(16k+11) = \varrho(16k+11) + \varrho(16k+10) - \varrho(8k+5).$$

Let $m \equiv 1 \pmod{16}$ be an arbitrary but fixed integer, and $16k+11$ coprime to m . Then from (4.3)

$$C + H(m(16k+11)) = \varrho(m(16k+11)) + \varrho(m(16k+11)-1) - \varrho\left(\frac{m(16k+11)-1}{2}\right),$$

and so

$$H(m) = \varrho(m(16k+11)) + \varrho(m(16k+11)-1) - \varrho\left(\frac{m(16k+11)-1}{2}\right) - \\ - \varrho(16k+11) - \varrho(16k+10) + \varrho(8k+5).$$

Hence we have

$$|H(m)| \sum_{\substack{k \leq x \\ (k, m)=1}} \frac{1}{k} \leq 6 \sum_{n \leq 18x} \frac{\varrho(n)}{n}.$$

Observing that

$$\lim_{x \rightarrow \infty} (\log x)^{-1} \sum_{\substack{k \leq x \\ (k, m)=1}} 1/k > 0,$$

and (1.3), we obtain that $H(m)=0$.

Let $m_1 \equiv m_2 \pmod{16}$ be odd integers. We can choose a v so that $(m_1 m_2, v)=1$ and $m_i v \equiv 1 \pmod{16}$. Then $H(m_1) + H(v) = H(m_2) + H(v) = 0$, whence $H(m_1) = H(m_2)$. We have that the value of $H(m)$ depends only on the residue class $m \pmod{16}$. But this is possible only if $H(m)=0$ for every odd n .

Let $n \equiv 1 \pmod{3}$. Then

$$(4.4) \quad \varrho(n) = \varrho(3n+2) + \varrho(3n+1) + \varrho(3n) - H(3n+2) - H(3n+1).$$

Let B_x denote the set of those integers n , for which

$$n \equiv 1 \pmod{3}, \quad 2^x \parallel 3n+1$$

hold. For $n \in B_x$ ($x \geq 1$) $H(3n+2)=0$. From (4.4) we get

$$|H(2^x)| \sum_{\substack{n \leq x \\ n \in B_x}} \frac{1}{n} \leq 3 \sum_{m \leq 4x} \frac{|\varrho(m)|}{m}.$$

Observing that

$$\lim_{x \rightarrow \infty} (\log x)^{-1} \sum_{\substack{n \leq x \\ n \in B_x}} \frac{1}{n} > 0,$$

from the relation (1.3) $H(2^x)=0$ follows. Consequently $H(n)=0$ identically, and so Theorem 1 is a consequence of Theorem 2.

References

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