

On (m, n) -ideals in associative rings

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1. Introduction

We introduce in this paper the following generalizations of the concepts of ideals in rings: the notion of (m, n) -ideal, which is a generalization of one-sided (left or right) ideals and it contains, as a particular case, the notion of bi-ideal due to R. A. GOOD and D. R. HUGHES [3] for semigroups, and due to the authors [17] for rings; the concept of (m, n) -quasi-ideal which is a generalization of the concept of quasi-ideal due to O. STEINFELD; the concept of i^k -ideal and k -ideal, the latter is a generalization of the concept of two-sided ideal.

Let us point out that (m, n) -ideals, (m, n) -quasi-ideals, i^k -ideals, and k -ideals were before introduced and investigated by the first author (see [6], [8], [12], [15]). For i^k -ideals, see also R. BAER [2].

For the necessary background notions used throughout this paper, we refer to N. JACOBSON [4], S. LAJOS [6], N. H. MCCOY [19], and F. SZÁSZ [23].

By a ring we mean an associative ring. Throughout this paper A always denotes an associative ring.

For the additive subgroups B and C of an associative ring A let the product BC denote the additive subgroup of the additive group A^+ of A , generated by all the product bc with $b \in B$ and $c \in C$. Let us remark that this complex multiplication of the additive subgroups is again associative, i.e. the relation

$$B(CD) = (BC)D$$

holds for arbitrary additive subgroups B, C, D of A^+ . Let furthermore $S^1 = S$ and $S^n = S^{n-1}S$ for an arbitrary subring S of A (S^0 is defined as an operator element such that $S^0A = AS^0 = A$). I always denotes the ring of rational integers and thus IX , for an arbitrary non-empty subset X of A , denotes the additive subgroup of A^+ generated by X .

In Section 2 we develop the fundamentals of the theory of (m, n) -ideals of associative rings. First we assert some important properties of (m, n) -ideals in rings. Furthermore, Theorem 2.4 gives a sufficient condition for a product BC , where B is an (m, n) -ideal and C is an additive subgroup of A^+ , to be again an (m, n) -ideal of A . In the same section we introduce the π -ideals of a ring, and we show in Theorem 2.7 that every lr -ideal is an $(1, 1)$ -ideal and conversely. More generally, our Theorem 2.9 asserts that an additive subgroup of A^+ is a π -ideal of A if and only if it is an (m, n) -ideal of A for certain non-negative integers m and n . Theorem 2.10 gives another sufficient condition for a product BC to be an (m, n) -ideal,

provided that B is an $(m, 0)$ -ideal and C is a $(0, n)$ -ideal of the ring A . Corollary 2.16 states that i^k -ideals and k -ideals coincide if the ring A is commutative. Finally, in Proposition 2.17 we point out that a ring A contains no proper (m, n) -ideals if and only if it is a division ring.

In Section 3 we discuss (m, n) -ideals of rings with minimum condition on subrings. Thus our Theorem 3.1 states that in the case $m, n \geq 2$ a ring A with minimum condition on subrings contains either a proper $(1, k)$ -ideal or else a proper $(k, 1)$ -ideal. Corollary 3.2 is the application of this result to finite rings.

In Section 4 we define the (m, n) -quasi-ideals of rings and investigate them. First we assert several fundamental properties. Then, in Theorem 4.2, we show that every (m, n) -quasi-ideal is an (m, n) -ideal of the ring. Theorem 4.4 asserts that in any ring with a distributive lattice of subrings, a subring is an (m, n) -quasi-ideal if and only if it is the intersection of an $(m, 0)$ -ideal and a $(0, n)$ -ideal of the ring.

In Section 5 we prove that in a von Neumann regular ring A every (m, n) -ideal is an (m, n) -quasi-ideal and conversely. Corollary 5.6 states that every i^k -ideal of a von Neumann regular ring A is a two-sided ideal of A .

2. (m, n) -ideals in associative rings

Definition 2.1. Let A be an associative ring, S a subring of A and m, n non-negative integers. We say that S is an (m, n) -ideal of A if $S^m A S^n \subseteq S$.

The present authors [17] defined the notion of $(1, 1)$ -ideal of rings under the name "bi-ideal" (cf. also for semigroups R. A. GOOD and D. R. HUGHES [3]).

It is easy to prove the following statements concerning (m, n) -ideals of rings.

a) The intersection of any two (m, n) -ideals of a ring A is again an (m, n) -ideal of A .

b) A division ring has no proper (m, n) -ideals.

c) The power S^k of an (m, n) -ideal S is again an (m, n) -ideal, where k is an arbitrary positive integer.

Let B be a non-empty subset of the ring A . Then the smallest (m, n) -ideal of A containing B is called the (m, n) -ideal of A generated by B and denoted by $\{B\}_{(m, n)}$.

d) It is clear that

$$\{B\}_{(m, n)} = IB + IB^2 + \dots + IB^{m+n} + IB^m A B^n.$$

e) If B is a subring of a ring A then

$$\{B\}_{(m, n)} = B + B^m A B^n.$$

f) The principal (m, n) -ideal of a ring A generated by the element a of A is the following:

$$\{a\}_{(m, n)} = Ia + Ia^2 + \dots + Ia^{m+n} + Ia^m A a^n.$$

g) The principal (m, n) -ideal of A generated by an idempotent element e is

$$\{e\}_{(m, n)} = \begin{cases} eA, & \text{if } m \neq 0, \quad n = 0; \\ eAe, & \text{if } m \neq 0, \quad n \neq 0; \\ Ae, & \text{if } m = 0, \quad n \neq 0. \end{cases}$$

Theorem 2.2. *Let A be a ring, S a subring of A and T an (m, n) -ideal of A . Then the intersection $S \cap T$ is an (m, n) -ideal of S .*

PROOF. The intersection $S \cap T$ obviously is a subring of S . We show that

$$(S \cap T)^m S (S \cap T)^n \subseteq S \cap T.$$

Since T is an (m, n) -ideal of A , we have

$$(S \cap T)^m S (S \cap T)^n \subseteq T.$$

On the other hand, S is a subring of A . Hence

$$(S \cap T)^m S (S \cap T)^n \subseteq S.$$

Therefore $S \cap T$ is an (m, n) -ideal of S .

Corollary 2.3. *Let B be an (m, n) -ideal, C a (p, q) -ideal of a ring A . Then the intersection $B \cap C$ is a (p, q) -ideal of the ring B and an (m, n) -ideal of the ring C .*

Theorem 2.4. *Let A be a ring, B an (m, n) -ideal of A and C an additive subgroup of A^+ satisfying either $BC \subseteq B$ or $CB \subseteq B$. Then both the products BC and CB are (m, n) -ideals of the ring A .*

PROOF. Assume, for instance, $BC \subseteq B$. Then the product BC is an additive subgroup of A^+ by the definition of BC . Furthermore we have

$$(BC)(BC) \subseteq B \cdot BC \subseteq BC,$$

thus the product BC is a subring of A . On the other hand,

$$(BC)^m A (BC)^n \subseteq B^m A B^{n-1} (BC) \subseteq BC,$$

whence BC is an (m, n) -ideal of the ring A .

It can similarly be proved that the product CB is also an (m, n) -ideal of the ring A .

Corollary 2.5. *Let m, n be positive integers, A a ring, S an (m, n) -ideal of A and $s \in S$. Then the products sS and Ss are (m, n) -ideals of A .*

Definition 2.6. A subring S_n of a ring A is said to be *attainable* if there exists a sequence $S_1, S_2, \dots, S_{n-1}$ of subrings of A such that

$$S_n \subseteq S_{n-1} \subseteq \dots \subseteq S_1 \subseteq S_0 = S$$

holds, where S_i is a one-sided (left or right) ideal of S_{i-1} ($i=1, 2, \dots, n$).

To every chain of such subrings of A we can associate a product π of the letters l and r whose i -th factor is l or r according to the fact that S_i is contained in S_{i-1} as a left or right ideal, respectively ($i=1, 2, \dots, n$). If S_i is a two-sided ideal of S_{i-1} , then any of l and r can be chosen. Furthermore, a subring S of A is called a π -ideal, if it is attainable by a subring chain to which the product π is associated. In the product π let m and n be the numbers of the factors r and l , respectively.

We are going to prove the following result.

Theorem 2.7. *For an additive subgroup S of a ring A the following statements are pairwise equivalent:*

- (i) S is an lr -ideal of A .
- (ii) S is an rl -ideal of A .
- (iii) S is an $(1, 1)$ -ideal of A .

PROOF. Suppose that S is an lr -ideal of a ring A . Then there exists a subring L of A such that $S \subseteq L \subseteq A$, $SL \subseteq S$, and $AL \subseteq L$. Hence

$$SAS \subseteq SAL \subseteq SL \subseteq S,$$

that is, S is an $(1, 1)$ -ideal of A , indeed.

Conversely, let S be an $(1, 1)$ -ideal of the ring A , that is,

$$SAS \subseteq S.$$

Then we have

$$S(S+AS) = S^2 + SAS \subseteq S + S = S.$$

It follows that S is a right ideal of the left ideal $S+AS$ of A . Consequently, S is an lr -ideal of A .

The proof of the left-right dual statement is analogous.

Corollary 2.8. *An additive subgroup S of a ring A is a π -ideal of A if and only if S is an $r^m l^n$ -ideal of A .*

This follows at once from the equivalence of the conditions (i), (ii) in Theorem 2.7.

Now we are ready to prove the following result.

Theorem 2.9. *An additive subgroup S of a ring A is a π -ideal of A if and only if S is an (m, n) -ideal of A .*

PROOF. By the preceding Corollary 2.8 it suffices to verify our statement for $r^m l^n$ -ideals instead of π -ideals. Let S be an $r^m l^n$ -ideal of a ring A . Then S is an attainable subring of A , that is, there exist subrings $L_1, L_2, \dots, L_{n-1}$ and $R_1, R_2, \dots, R_m$ of A such that the following relations hold:

$$S = L_n \subseteq L_{n-1} \subseteq \dots \subseteq L_1 \subseteq R_m \subseteq \dots \subseteq R_1 \subseteq R_0 = A,$$

$$R_i R_{i-1} \subseteq R_i, \quad R_m L_1 \subseteq L_1, \quad L_{j-1} L_j \subseteq L_j$$

$$(i = 1, 2, \dots, m; j = 1, 2, \dots, n).$$

This yields at once

$$\begin{aligned} S^m A S^n &= L_n^m A L_n^n \subseteq L_n^{m-1} (R_1 A) L_n^n \subseteq L_n^{m-1} R_1 L_n^n \subseteq L_n^{m-2} (R_2 R_1) L_n^n \subseteq L_n^{m-2} R_2 L_n^n \subseteq \\ &\subseteq \dots \subseteq (L_n R_{m-1}) L_n^n \subseteq (R_m R_{m-1}) L_n^n \subseteq R_m L_n^n \subseteq \\ &\subseteq (R_m L_1) L_n^{n-1} \subseteq L_1 (L_2 L_n^{n-2}) \subseteq \dots \subseteq L_n = S, \end{aligned}$$

therefore S is an (m, n) -ideal of the ring A , indeed.

Conversely, let us suppose that S is an (m, n) -ideal of a ring A . Then, by the property $e)$, the (p, q) -ideal of A generated by S is $S + S^p A S^q$. Now it is easy to see, that $S_{(m, k)}$ is a left ideal of $S_{(m, k-1)}$ ($k = 1, \dots, n$), and $S_{(i, n)}$ is a right ideal of

$S_{(i-1, n)}$ ($i=1, \dots, m$). Hence the subrings $L_n=S$, $L_{n-1}=S_{(m, n-1)}$, $\dots$, $L_1=S_{(m, 1)}$, $R_m=S_{(m, 0)}$, $R_{m-1}=S_{(m-1, 0)}$, $\dots$, $R_1=S_{(1, 0)}$ satisfy the above conditions. Thus S is an r^m/n -ideal of the ring A , which completes the proof of Theorem 2.9.

Theorem 2.10. *Let m, n be positive integers, B an $(m, 0)$ -ideal and C a $(0, n)$ -ideal of a ring satisfying the condition $BC=CB$. Then the product BC is an (m, n) -ideal of A .*

PROOF. Obviously, the condition $BC=CB$ yields

$$(BC)^2 = B^2C^2 \subseteq BC,$$

thus the product BC is a subring of A . On the other hand, we have

$$(BC)^mA(BC)^n = B^m(C^mAB^n)C^n \subseteq (B^mA)C^n \subseteq BC,$$

consequently BC is an (m, n) -ideal of A , indeed.

Remark 2.11. In the particular case $m=n=1$ the assumption $BC=CB$ is superfluous. Namely, if L is a left ideal and R is a right ideal of a ring A , then the products LR and RL are $(1, 1)$ -ideals of A . (Moreover, LR is a two-sided ideal of A .)

Definition 2.12. A two-sided ideal of a two-sided ideal of a ring A will be called an i^2 -ideal of A . By an i^k -ideal we mean a two-sided ideal of an arbitrary i^{k-1} -ideal of A , where k is a positive integer ($k \geq 2$).

Remark 2.13. The i^k -ideals for $k=2, 3, 4, \dots$ are called also accessible subrings. Furthermore, R. BAER [2] calls these subrings "metaideals of finite index".

Definition 2.14. The subring S of a ring A is called a k -ideal of A , if S is an (m, n) -ideal of A for every m, n such that $m+n=k$.

Remark 2.15. It is clear that a subring S of a commutative ring A is a k -ideal of A if and only if the condition

$$S^kA \subseteq S$$

holds. We observe that the concept of the k -ideal is a generalization of the concept of two-sided ideal.

Corollary 2.16. *A subring S of a commutative ring A is an i^k -ideal of A if and only if it is a k -ideal of A .*

PROOF. This follows at once from our Theorem 2.9.

Proposition 2.17. *A ring A contains no proper (m, n) -ideal if and only if it is a division ring.*

PROOF. The sufficiency is obvious. The necessity proof is the following. For every element $a \in A$ the subrings aA and Aa are (m, n) -ideals of A . Therefore

$$aA = Aa = A,$$

which implies that A is a division ring.

3. (m, n) -ideals in rings with minimum condition on subrings

As it is well known, the rings with minimum condition on subrings have been studied by V. I. ŠNEJDMÜLLER [21].

Next we prove the following result.

Theorem 3.1. *Let A be a ring with minimum condition on subrings. If A contains a proper (m, n) -ideal with $m \geq 2$ and $n \geq 2$, then A contains also a proper $(1, k)$ -ideal or a proper $(k, 1)$ -ideal for some $k \geq 2$.*

PROOF. Let m_1 denote the smallest positive integer for which A contains a proper (m_1, n) -ideal, and n_1 the smallest positive integer for which A contains a proper (m, n_1) -ideal. Then, by our assumptions, $m, n, m_1, n_1 \geq 2$. We show that either $m_1 \leq n$ or $n_1 \leq m$ holds. Suppose that, in contrary, we have both of $m_1 > n$ and $n_1 > m$. Then, by $m_1 \leq m$, we obtain $n_1 > m \geq m_1 > n$, which is a contradiction with respect to the definition of n_1 .

Thus we may assume, for instance, that $2 \leq m_1 \leq n$. Furthermore, let S be a proper (m_1, n) -ideal of the ring A . Now we define the following sequence of subrings of A :

$$B_1 = S^m A S^n, \quad B_{i+1} = B_i^{m_1} A B_i^n \quad (i = 1, 2, \dots).$$

We can verify that $B_i^2 \subseteq B_i$. It is easy to see that

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \quad \text{and} \quad B_i^{m_1} A B_i^n \subseteq S.$$

Now the minimum condition on subrings of A implies

$$B_j = B_{j+1} = B_{j+2} = \dots = B$$

for a fixed positive integer j . Then we have

$$B = B^{m_1} A B^n,$$

whence it follows at once

$$B^{m_1} A B^{n-m_1} B^{m_1} A B^n = B A B^n.$$

Therefore we obtain

$$B^{m_1} A B^{n-m_1+1} = B A B^n$$

as well. Postmultiplying this equality by B^{m_1-1} , we arrive

$$B^{m_1} A B^{n-m_1+1} B^{m_1-1} = B A B^n \cdot B^{m_1-1}.$$

Finally we obtain

$$B = B A B^{n+m_1-1}.$$

In other words, the subring B of A is an $(1, k)$ -ideal of A with $k = n + m_1 - 1$.

One can prove in a similar way, that in the case of $n_1 \leq m$ there exists a proper $(k, 1)$ -ideal of the ring A .

Corollary 3.2. *If a finite ring A contains a proper (m, n) -ideal with $m \geq 2$ and $n > 2$, then A contains also a proper $(1, k)$ -ideal, or else a proper $(k, 1)$ -ideal of A .*

4. (m, n) -quasi-ideals of rings

Definition 4.1. A subring S of a ring A will be called an (m, n) -quasi-ideal of A if the inclusion

$$S^m A \cap A S^n \subseteq S$$

holds, where m, n are non-negative integers (S^0 is an operator element not contained in A and such that $S^0 A = A S^0 = A$).

It is easy to prove the following properties of (m, n) -quasi-ideals:

a) The intersection of a set of (m, n) -quasi-ideals of a ring A is again an (m, n) -quasi-ideal of A .

b) A division ring has no proper (m, n) -quasi-ideal.

c) Let B be a subset of a ring A . Then the (m, n) -quasi-ideal of A generated by B , i.e., the smallest (m, n) -quasi-ideal of A containing B is

$$IB + IB^2 + \dots + IB^k + (IB^m A \cap IAB^n),$$

where $k = \max(m, n)$.

d) If S is a subring of a ring A , then the (m, n) -quasi-ideal of A generated by S is

$$S + (S^m A \cap A S^n).$$

e) The principal (m, n) -quasi-ideal generated by the element a of a ring A is

$$Ia + Ia^2 + \dots + Ia^k + (a^m A \cap Aa^n),$$

where $k = \max(m, n)$.

f) The principal (m, n) -quasi-ideal generated by an idempotent element e of A is

$$e^m A \cap A e^n = \begin{cases} Ae & \text{if } m = 0, \quad n \neq 0; \\ eA \cap Ae & \text{if } m \neq 0, \quad n \neq 0; \\ eA & \text{if } m \neq 0, \quad n = 0. \end{cases}$$

Theorem 4.2. Every (m, n) -quasi-ideal of a ring A is an (m, n) -ideal of A .

PROOF. Let A be a ring and S an (m, n) -quasi-ideal of A . Since we have $S^m A S^n \subseteq S^m A$ and $S^m A S^n \subseteq A S^n$, we obtain

$$S^m A S^n \subseteq S^m A \cap A S^n \subseteq S,$$

that is, S is an (m, n) -ideal of the ring A .

Remark 4.3. The concept of $(1, 1)$ -quasi-ideal was introduced by O. STEINFELD [22] under the name "quasi-ideal". He showed that a subset of a semigroup is a quasi-ideal if and only if it is the intersection of a left ideal and a right ideal. The corresponding problem for associative rings is yet open. It has been answered for non-associative rings in the negative. (Cf. M. SADIQ ZIA, Studies in ring theory, Dissertation, L. Eötvös University, Budapest, 1975.)

Now we prove the following

Theorem 4.4. Assume that the subrings of a ring A constitute a distributive lattice with respect to addition and intersection. Then a subring of the ring A is an (m, n) -quasi-ideal of A if and only if it is the intersection of an $(m, 0)$ -ideal and a $(0, n)$ -ideal of A .

PROOF. Let A be a ring with distributive subring lattice, let B and C be an $(m, 0)$ -ideal and a $(0, n)$ -ideal of A , respectively. Then we have $B^m A \subseteq B$ and $AC^n \subseteq C$, whence

$$(B \cap C)^m A \cap A (B \cap C)^n \subseteq B \cap C,$$

and since the common part of subrings is again a subring, we obtain that the intersection $B \cap C$ is an (m, n) -quasi-ideal of A .

Conversely, suppose that S is an (m, n) -quasi-ideal of A . Then we have $S^m A \cap AS^n \subseteq S$. We prove that

$$S = \{S\}_{(m, 0)} \cap \{S\}_{(0, n)}$$

provided that the subring lattice of A is distributive. By the property $e)$ of Section 2, we have

$$\{S\}_{(m, 0)} = S + S^m A,$$

and

$$\{S\}_{(0, n)} = S + AS^n.$$

The assumed distributivity of the subring lattice yields

$$(S + S^m A) \cap (S + AS^n) = S + (S^m A \cap AS^n) = S,$$

as we stated.

5. (m, n) -ideals in von Neumann regular rings

Definition 5.1. A ring A is called *regular* (in the sense of J. VON NEUMANN), if to every element a of A there exists an element x in A such that $axa = a$.

Theorem 5.2. In a von Neumann regular ring A every (m, n) -ideal S is an (m, n) -quasi-ideal of A and conversely.

PROOF. Let A be a von Neumann regular ring. We show that

$$S^m AS^n = S^m A \cap AS^n$$

if S is an (m, n) -ideal of A . From the proof of Theorem 4.2 we know that $S^m AS^n \subseteq S^m A \cap AS^n$. Conversely, the regularity criterion of L. KOVÁCS [5] implies

$$S^m A \cap AS^n \subseteq (S^m A)(AS^n) \subseteq S^m AS^n.$$

Corollary 5.3. In a von Neumann regular ring A every bi-ideal of A is a quasi-ideal of A (and conversely).

For this corollary, see S. LAJOS [7].

In what follows we need the following well known lemma of V. A. ANDRUNAKIEVIČ [1].

Lemma 5.4. Suppose that B is an ideal of a ring A and C is an ideal of the ring B . Let C^* be the ideal of A generated by C . Then

$$(C^*)^3 \subseteq C.$$

A consequence of Lemma 5.4 reads as follows.

Corollary 5.5. *If every two-sided ideal of a ring A is idempotent then every i^k -ideal of A is a two-sided ideal of A .*

PROOF. The assumption and Lemma 5.4 imply

$$(C^*)^3 \subseteq C \subseteq C^*$$

and $(C^*)^2 = C^*$, that is, $C = C^*$ holds, which completes the proof of Corollary 5.5.

Corollary 5.6. *In a von Neumann regular ring A every i^k -ideal is a two-sided ideal of A .*

PROOF. Evidently, every one-sided (left or right) ideal of a von Neumann regular ring is idempotent. Thus Corollary 5.5 implies Corollary 5.6.¹⁾

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