

Notes on matrix-valued stationary stochastic processes I

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1. Spectral representation

1.0. The spectral representation of the matrix-valued stationary stochastic process (m. st. p.) is given in this study. The m. st. p. — in quadratic case — is defined by T. BALOGH in [1]. Here the "matrix-valued random variable" denotes a $p \times q$ -type matrix the elements of which are complex random variables. The values of

p and q are $1, 2, 3, \dots (< \infty); 1, 2, 3, \dots$

respectively. The discrete m. st. p. will be examined when the values of p and q are arbitrarily chosen constants. Specifically it means

- a p -dimension vector ($p \geq 1; q = 1$)
- a $p \times p$ -type quadratic matrix ($p = q$)
- a $p \times q$ -type rectangle matrix ($p \neq q; q < \infty$)
- a ribbon-matrix ($p \geq 1; q = +\infty$) valued discrete process.

1.1. M_p denotes the set of the $p \times p$ -type quadratic matrices with complex elements.

$\mathcal{M}_2^{p \times q}$ denotes the set of the $p \times q$ -type matrix-valued random variables, the elements of which have got variance, if $q = \infty$ the (1.1) is required too.

If $\xi \in \mathcal{M}_2^{p \times q}$ then $E\xi$ denotes the expectation matrix, that is the matrix of the expectations of the elements. For every $\xi, \eta \in \mathcal{M}_2^{p \times q}$ there are their covariance matrix

$$\text{cov}(\xi, \eta) = \langle \xi - E\xi, \eta - E\eta \rangle = E(\xi - E\xi)(\eta - E\eta)^* \in M_p$$

(* denotes, as usual the conjugate of the transpose of matrix.) The

$$\{\xi_n | \xi_n \in \mathcal{M}_2^{p \times q}, E\xi_n = 0, n = 0, \pm 1, \pm 2, \dots\} = \{\xi_n\}_{-\infty}^{+\infty}$$

is m. st. p. if

$$\text{cov}(\xi_n, \xi_m) = \Gamma(n - m)$$

for every n, m .

The $\{\xi_n\}_{-\infty}^{+\infty}$ m. st. p. is a curve in the $\mathcal{M}_2^{p \times q}$ Q.H. space. The concept of Q.H. — Quasi Hilbert — space is given by B. GYIRES in [2].

(The inner product is in $\mathcal{M}_1^{p \times q}$: $\langle \xi, \eta \rangle = E\xi\eta^*$).

For $\xi \in \mathcal{M}_2^{p \times q}$ the ξ^i denotes the vector-valued random variable determined by the i -th row of the matrix, thus $\xi = [\xi^i]_{i=1}^p$.

If $q = \infty$ then $\underline{\xi}^i$ is an infinite-dimensional row vector, and the

$$(1.1) \quad E \underline{\xi}^i \underline{\xi}^{i*} < \infty$$

condition is required.

1.2. Let us denote the Q. H. space generated by m. st. p. $\{\xi_n\}_{-\infty}^{\infty}$ and the Hilbert space generated by row vectors of this process with

$$Q_{\infty} = Q(\{\xi_n\}_{-\infty}^{\infty}), \quad v_{\infty} = v(\{\xi_n^i (i = 1, 2, \dots, p)\}_{-\infty}^{\infty})$$

respectively.

1.1. Lemma. *The elements of the Q_{∞} Q. H. space are built by exactly those row vectors which are the elements of the v_{∞} Hilbert space.*

The state of 1.1. lemma follows obviously from the constructions of Q_{∞} and v_{∞} spaces.

Now we can define an U unitary operator on the $\{\xi_n^i, i = 1, 2, \dots, p\}_{-\infty}^{\infty}$ subset of v_{∞} ,

$$U \xi_n^i = \xi_{n+1}^i.$$

This U unitary operator based on 4.1. lemma of [3]. (p. 14) can be extended over v_{∞} and it will be unitary, too.

After that let V by definition

$$V \xi = [U \xi]_{i=1}^p, \quad \xi \in Q_{\infty}$$

an unitary operator on Q_{∞} Q. H. space.

Properties of V are:

$$a) \quad V \xi_n = V^{n+1} \xi_0 = \xi_{n+1}$$

$$b) \quad V = [U] = \left[\int_0^{2\pi} e^{i\lambda} dF(\lambda) \right]$$

(see 109 theorem of [4]).

The $\{\xi_n\}_{-\infty}^{\infty}$ m. st. p. can be represented by

$$(1.2) \quad \xi_n = V^n \xi_0 = \left[\int_0^{2\pi} e^{in\lambda} d(E(\lambda) \xi_0^j) \right]_{j=1}^p = \int_0^{2\pi} e^{in\lambda} dZ(\lambda)$$

where $Z(\lambda) = [E(\lambda) \xi_0^j]_{j=1}^p \in \mathcal{M}_2^{p \times q}$ is an orthogonal stochastical measure defined on the interval $[0, 2\pi]$ having the properties

a) σ -additivity

b) for $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \leq 2\pi$

$$\langle Z(\lambda_2) - Z(\lambda_1), Z(\lambda_4) - Z(\lambda_3) \rangle = 0 (\in M_p).$$

After (1.2) spectral representation of the m. st. p. we can construct the Herglotz-representation of the $\Gamma(\cdot)$ covariance function:

$$(1.3) \quad \Gamma(n) = \langle \xi_k, \xi_{k+n} \rangle = \int_0^{2\pi} e^{in\lambda} dF(\lambda)$$

where $F(\lambda) = \langle Z(\lambda), Z(\lambda) \rangle$ is a matrix-valued function uniquely determined, non-negative definite, Hermitian-symmetric, monotonically non-decreasing, of bounded variation in its elements and defined on the interval $[0, 2\pi]$ referred to as spectral distribution function of the m. st. p. $\{\xi_n\}_{-\infty}^{\infty}$

Bibliography

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