

On a numbertheoretical series

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A composite number n is called a *pseudoprime* if $n|2^n-2$.

Let $P(x)$ denote the number of pseudoprimes $\leq x$ and P_n the n -th pseudoprime. In 1949 P. ERDŐS stated that

$$(1) \quad c_1 \log x < P(x) < c_2 x/(\log x)^k, \quad \text{for every } k \text{ and } x > x_0(k).$$

K. SZYMICZEK [10] proved, using the following result of P. ERDŐS (see [2])

$$(2) \quad P(x) < 2x \exp \left\{ -\frac{1}{3} (\log x)^{1/4} \right\} \quad \text{if } x > x_0$$

that $1/P_n < 2/n(\log n)^{4/3}$. Therefore $\sum_{n=1}^{\infty} 1/P_n < \sum_{n=1}^{\infty} 2/n(\log n)^{4/3}$ and since the last series is convergent $\sum_{n=1}^{\infty} 1/P_n$ is also convergent. A. ROTKIEWICZ [9] proved that

$$(3) \quad P(x) > \frac{5}{8} \log_2 x \quad \text{for } x \geq 1905, \quad \text{where } \log_2 x \text{ denotes}$$

the logarithm at the base 2. This result is much stronger than the theorem of K. SZYMICZEK [10]:

$$(4) \quad P(x) > \frac{1}{4} \{ \log x + \log \log x + \dots + \underbrace{\log \log \dots \log x}_{k \text{ times}} \}$$

ROTKIEWICZ ([7], problem 47) asked whether the series $\sum 1/\log P_n$ is convergent. A. MAKOWSKI [5] proved that the series $\sum 1/\log P_n(c)$ is divergent, where $P_n(c)$ denotes the n -th pseudoprime with respect to c (n is a pseudoprime with respect to c if n is composite and $n|c^n-c$). He used the fact established by CIPOLLA [1] that the number $(c^{2p}-1)/(c^2-1)$ is a pseudoprime with respect to c if p is an odd prime such that $p \nmid c^2-1$ and that the series $\sum 1/p$, where p runs over the primes, is divergent.

First we note that the divergence of $\sum_{n=1}^{\infty} 1/\log P_n$ follows from the estimation $P(x) > c \log x$. Indeed, if we put $x = P_n$ we get

$$(5) \quad P(P_n) > c \log P_n$$

and since $P(P_n)=n$, by (5)

$$\frac{1}{\log P_n} > c/n$$

and the divergence follows at once from the well-known divergence of the series $\sum_{n=1}^{\infty} 1/n$.

Theorem 1. *The series $\sum 1/\log p_n^{(a)}$, where $p_n^{(a)}$ is the n -th pseudoprime of the form $ax+b$, where $(a,b)=1$, is divergent.*

PROOF. Let a, b be fixed coprime positive integers. Let $P^{(a)}(x)$ denote the number of pseudoprimes $\equiv b \pmod{a}$ and $\leq x$. By theorem 5 of A. ROTKIEWICZ [8] we have that

$$(6) \quad P^{(a)}(x) \gg \log x / (a^c \log \log x),$$

where c is an absolute constant.

If we put $x=P_n^{(a)}$ in (6) we get

$$(7) \quad n \gg \log P_n^{(a)} / \log \log P_n^{(a)}$$

and hence $\log \log P_n^{(a)} \ll \log n$. Thus by (7)

$$(8) \quad \log P_n^{(a)} \ll n(\log \log P_n^{(a)}) \ll n \log n.$$

Hence it follows that

$$(9) \quad \sum 1/\log P_n^{(a)} \gg \sum 1/n \log n$$

and the divergence of the first series in (9) follows from the well-known divergence of $\sum 1/n \log n$. This completes the proof of Theorem 1.

Theorem 2. *If $n \geq 7$, then*

$$e^{\log n + c \sqrt{(\log n)(\log \log n)}} < P_n < e^{\frac{8}{5} n \log 2}$$

where c is an absolute positive constant.

PROOF. In 1955 P. ERDŐS [4] proved with Knödel's method (see [6]) that,

$$(10) \quad P(x) < x e^{-c \sqrt{(\log x)(\log \log x)}}$$

where c is an absolute positive constant. Put $x=P_n$, then by (10)

$$(11) \quad P_n > n e^{c \sqrt{(\log P_n)(\log \log P_n)}} > n e^{c \sqrt{(\log n)(\log \log n)}}.$$

This completes the proof of the left side of the inequality. On the other hand by (3) with $x=P_n$, $P_7=1905$,

$$(12) \quad n > \frac{5}{8} \log_2 P_n \quad \text{if } n \geq 7,$$

and so clearly

$$(13) \quad e^{\frac{8}{5}n \cdot \log 2} > P_n \quad \text{if } n \geq 7.$$

This completes the proof.

Theorem 3. *The series $\sum_{n=1}^{\infty} e^{c\sqrt{(\log P_n)(\log \log P_n)}}/P_n(\log n)^s$ is convergent for $s > 1$ (c is the constant occurring in the Erdős result).*

PROOF. By (11)

$$P_n > ne^{c\sqrt{(\log P_n)(\log \log P_n)}}$$

hence

$$1/n(\log n)^s > e^{c\sqrt{(\log P_n)(\log \log P_n)}}/P_n(\log n)^s$$

and so the theorem follows from the well-known convergence of the series

$\sum_{n=1}^{\infty} 1/(n(\log n)^s)$ for $s > 1$. P. ERDŐS [4] conjectured that

$$(14) \quad P(x) > x^{1-\varepsilon} \quad \text{for every } \varepsilon > 0 \quad \text{and } x > x_0(\varepsilon) \quad \text{and}$$

gave strong reasons for this conjecture.

Theorem 4. *From the conjecture of P. Erdős it follows that the series $\sum 1/P_n^{1-\varepsilon}$ is divergent for every $\varepsilon > 0$.*

PROOF. Suppose that $\varepsilon > 0$ then by the conjecture of Erdős:

$$(15) \quad P(x) > x^{1-\varepsilon} \quad \text{for } x > x_0(\varepsilon).$$

It follows that

$$(16) \quad n > P_n^{1-\varepsilon} \quad \text{for } n > n_0(\varepsilon)$$

and hence

$$(17) \quad \sum 1/P_n^{1-\varepsilon} \gg \sum 1/n$$

and it follows that the series $\sum 1/P_n^{1-\varepsilon}$ is divergent. We have seen that the series $\sum 1/\log P_n$ is divergent.

If we could prove that $P(x) \gg (\log x)^k$, then it would follow that the series $\sum 1/(\log P_n)^k$ is divergent. However this result seems hard to get.

Remark. From Theorem 2 it follows that

$$P_n > n^{1+c\sqrt{(\log \log n)/\log n}},$$

where c is an absolute positive constant.

On the other hand from the conjecture of P. Erdős it follows that $P(x) > x^{1-\varepsilon}$ for all $\varepsilon > 0$ and all $x > x_0(\varepsilon)$ and so

$$P_n < n^{\frac{1}{1-\varepsilon}} = n^{1+\frac{\varepsilon}{1-\varepsilon}} \quad \text{for } n > n_0(\varepsilon).$$

Thus for all $\bar{\varepsilon} > 0$, $P_n < n^{1+\bar{\varepsilon}}$ if $n > n_0(\bar{\varepsilon})$ and hence if Erdős' conjecture is true the following upper and lower bounds of P_n holds provided that n is sufficiently large

$$(18) \quad n^{1+c\sqrt{(\log \log n)/\log n}} < P_n < n^{1+\bar{\varepsilon}},$$

where c is an absolute positive constant.

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