On a Q-recurrent Finsler space of second order

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1. Introduction

Let Fn be an n-dimensional Finsler space equipped with positively homogeneous metric function $F(x, \dot{x})$ of degree one in its directional arguments and satisfies the requisite conditions [1][1]) imposed upon it. The fundamental metric tensor of the space is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \quad (\dot{\partial}_i \equiv \partial/\partial \dot{x}^i). \tag{1.1}$$

The projective covariant derivative [2] of a vector field $X^i(x, \dot{x})$ with respect to x^k is given by

$$x^i_{((k))} = \partial_k x^i - (\dot{\partial}_m x^i) \Pi^m_{rk} \dot{x}^r + x^h \Pi^i_{hk} \tag{1.2}$$

where the projective connection coefficients

$$\Pi^i_{hk}(x, \dot{x}) \overset{\text{def}}{=} G^i_{hk} - \frac{1}{(n+1)} (2\delta^i_{(h} G^r_{k)r} + \dot{x}^i G^r_{rkh}) \tag{1.3}$$

are positively homogeneous of degree zero in $\dot{x}^i$'s and satisfy the following relations:

$$\text{a) } \Pi^i_{hkr} \dot{x}h = 0 \quad \text{and} \quad \text{b) } \dot{\partial}_h \Pi^i_{jk} = \Pi^i_{hjk}. \tag{1.4}$$

In particular the projective covariant derivative of x^i vanishes i.e.

$$\dot{x}^i_{((k))} = 0. \tag{1.5}$$

The commutation formulae involving the projective covariant derivative of tensor field $T^i_j(x, \dot{x})$ are given by

$$\dot{\partial}_h (T^i_{j((k))}) - (\dot{\partial}_h T^i_j)_{((k))} = T^s_j \Pi^i_{shk} - T^i_s \Pi^s_{jhk} \tag{1.6}$$

and

$$2T^i_{j[((h))((k))]} = -(\dot{\partial}_r T^i_j) Q^r_{shk} \dot{x}^s + T^s_j Q^i_{shk} - T^i_s Q^s_{jhk}, \tag{1.7}$$

where the projective entity $Q^i_{hjk}(x, \dot{x})$ is given by

$$Q^i_{hjk}(x, \dot{x}) = 2 \{ \partial_{[k} \Pi^i_{j]h} - (\dot{\partial}_r \Pi^i_{h[j}) \Pi^r_{k]s} \dot{x}^s + \Pi^r_{h[j} \Pi^i_{k]r} \} \tag{1.8}$$

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[1]) The numbers in square brackets refer to the references given at the end of the paper.

and satisfies the following relation [2]

$$Q^i_{hk} = Q^i_{jhk}\dot{x}^j. \tag{1.9}$$

The first order recurrency condition for the projective entity $Q^i_{hjk}(x, \dot{x})$ is given by

$$Q^i_{hjk((l))} = u_l Q^i_{hjk}, \tag{1.10}$$

where $u_l(x)$ is a recurrence vector depending only upon positional coordinates.

Transvecting (1.10) by $\dot{x}^h$ in view of equations (1.5) and (1.9), we get

$$Q^i_{jk((l))} = u_l Q^i_{jk}. \tag{1.11}$$

2. Q-recurrent Finsler space of second order

Definition (2.1): In an Fn if the projective Q^i_{hjk} satisfies the relations

$$Q^i_{hjk((l))((m))} = b_{lm} Q^i_{hjk} \tag{2.1}$$

and

$$Q^i_{hjk} \neq 0, \tag{2.2}$$

where $b_{lm}(x, \dot{x})$ is a tensor field depending both upon positional and directional arguments, then it is said to be Q-recurrent Finsler space of second order and b_{sm} is called a recurrent tensor field. In this paper we shall denote such a Finsler space by Fn^*.

Theorem (2.1). *A relation between $u_l(x)$ and $b_{lm}(x, \dot{x})$ is given by*

$$b_{lm} = u_{l((m))} + u_l u_m. \tag{2.3}$$

Proof. Differentiating (1.10) projective covariantly with respect to x^m and using equations (1.10) itself and (2.1), we get the required result (2.3).

Theorem (2.2). *In an Fn^* the recurrence tensor field $b_{lm}(x, \dot{x})$ is non-symmetric.*

Proof. Commutating (2.1) with respect to the indices s and m we get

$$2Q^i_{hjk[((l))((m))]} = (b_{lm} - b_{ml}) Q^i_{hjk}. \tag{2.4}$$

In view of the commutation formula (1.7) the above equation reduces to

$$\begin{aligned}(b_{lm} - b_{ml}) Q^i_{hjk} = &-(\dot{\partial}_r Q^i_{hjk}) Q^r_{lm} + Q^r_{hjk} Q^i_{rlm} - \\ &- Q^i_{rjk} Q^r_{hlm} - Q^i_{hrk} Q^r_{jlm} - Q^i_{hjr} Q^r_{klm}\end{aligned} \tag{2.5}$$

which proves the theorem.

Theorem (2.3). *In an Fn^* if u_s is independent of $\dot{x}^i$ the recurrence tensor field $b_{sm}(x, \dot{x})$ satisfies the following relation*

$$\dot{x}^s \{(b_{lm} - b_{ml})_{((s))} - u_s (b_{lm} - b_{ml})\} = 0. \tag{2.6}$$

PROOF. In view of the commutation formula (1.6) differentiating (2.5) projective covariantly with respect to x^s and using the equations (1.10), (1.11), (2.5) itself and the fact that u_s is independent of $\dot{x}^i$, we have

$$(b_{lm}-b_{ml})_{((s))}Q^i_{hjk} = u_s(b_{lm}-b_{ml})Q^i_{hjk} + + Q^r_{lm}\{Q^p_{hjk}\Pi^i_{prs} - Q^i_{pjk}\Pi^p_{hrs} - Q^i_{hpk}\Pi^p_{jrs} - Q^i_{hjp}\Pi^p_{krs}\}. \tag{2.7}$$

Transvecting (2.7) by $\dot{x}^s$ and noting equation (1.4a), we obtain

$$Q^i_{hjk}\{(b_{lm}-b_{ml})_{((s))} - u_s(b_{lm}-b_{ml})\}\dot{x}^s = 0 \tag{2.8}$$

which in view of the assumption that $Q^i_{hjk} \neq 0$, gives the required result.

Theorem (2.4). *In an Fn^* the recurrence tensor field satisfies the following relation:*

$$\dot{x}^s\{(b_{lm}-b_{ml})_{((s))((n))} - u_{s((n))}(b_{lm}-b_{ml}) - 2(b_{lm}-b_{ml})_{[((n))}u_{s]}\}. \tag{2.9}$$

PROOF. Differentiating (2.7) projective covariantly with respect to x^n and using (1.10), (1.11), (2.3) and (2.7) itself, we get

$$Q^i_{hjk}[(b_{lm}-b_{ml})_{((s))((n))} - u_{s((n))}(b_{lm}-b_{ml}) - 2(b_{lm}-b_{ml})_{[((n))}u_{s]}] - - Q^r_{lm}\{Q^p_{hjk}\Pi^i_{prs((n))} - Q^i_{pjk}\Pi^p_{hrs((n))} - Q^i_{hpk}\Pi^p_{jrs((n))} - Q^i_{hjp}\Pi^p_{krs((n))}\} = 0. \tag{2.10}$$

Transvecting (2.10) by $\dot{x}^s$ and noting the equations (1.4a) and (1.5), we get

$$Q^i_{hjk}[(b_{lm}-b_{ml})_{((s))((n))} - u_{s((n))}(b_{lm}-b_{ml}) - 2(b_{lm}-b_{ml})_{[((n))}u_{s]}]\dot{x}^s = 0 \tag{2.11}$$

which in view of the assumption (2.2) yields the required result.

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