

A note on non existence of interpolatory polynomials for (0, 2)-interpolation

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Let there be given n points $x_1, x_2, \dots, x_n$ such that

$$(1) \quad -1 \leq x_n < x_{n-1} < \dots < x_2 < x_1 \leq 1$$

and let $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ be arbitrary real numbers. In 1955 TURÁN and his associates [10] wanted to find out whether or not there is a polynomial $R_n(x)$ of degree $\leq 2n-1$ such that

$$(2) \quad \begin{cases} R_n(x_v) = \alpha_v \\ R_n''(x_v) = \beta_v \end{cases} \quad (v = 1, 2, 3, \dots, n).$$

They termed this kind of interpolation as (0, 2)-interpolation. By choosing x_v 's as the zeros of $\pi_n(x) = (1-x^2)P_{n-1}'(x)$, where $P_{n-1}(x)$ stands for the $(n-1)^{th}$ degree Legendre polynomial, they were able to show that if n is odd then the polynomials $R_n(x)$ do not exist in general. But they are uniquely determined when n is even. In other words if n is even and $\alpha_v = \beta_v = 0$, $v = 1, 2, \dots, n$ then the only polynomial of degree $\leq 2n-1$ satisfying (2) is identically zero. They also obtained the explicit representation of these polynomials and proved a uniform convergence theorem [1], [2]. Later this uniform convergence theorem was improved by FREUD [3]. Since then many investigators ([4], [5], [6], [8], [9], [13]) have obtained such results by taking the roots of $T_n(x)$, $H_n(x)$, $xL_n'(x)$ and $(1-x^2)P_n^{(\alpha, -\alpha)}(x)$ as nodes of interpolation where $T_n(x)$, $H_n(x)$, $L_n(x)$ and $P_n^{(\alpha, -\alpha)}(x)$ are Tchebychev polynomial of first kind, Hermite polynomial, Laguerre polynomial and Jacobi polynomial with $(\alpha, \beta) = (\alpha, -\alpha)$, of degree n respectively.

Let

$$(3) \quad -1 = x_{n+2} < x_{n+1} < \dots < x_2 < x_1 = 1$$

be the zeros of $w_n(x) = (1-x^2)P_n(x)$, where $P_n(x)$ is the Legendre polynomial of degree n . In [7] we proved that there exists a unique polynomial $R_n(x)$ of degree $\leq 2n+1$ satisfying the conditions:

$$(4) \quad \begin{aligned} R_n(x_v) &= \alpha_v; & v &= 1, 2, \dots, n+2 \\ R_n''(x_v) &= \beta_v; & v &= 2, 3, \dots, n+1 \end{aligned}$$

provided n is an even positive integer where $\{\alpha_v\}_1^{n+2}$ and $\{\beta_v\}_2^{n+1}$ are any preassigned values. But for n odd there does not exist, in general, a unique polynomial of

degree $\leq 2n+1$ satisfying the conditions of (4). Recently Varma [12] has proved the following:

Theorem 1. *Given a positive integer n and real numbers*

$$\alpha_k (k = 1, 2, \dots, n+2), \quad \beta_k, \gamma_k (k = 2, 3, \dots, n+1)$$

there is, in general, no polynomial $P_{3n+1}(x)$ of degree $\leq 3n+1$ such that

$$(5) \quad P_{3n+1}(x_k) = \alpha_k; \quad k = 1, 2, \dots, n+2,$$

$$(6) \quad P'_{3n+1}(x_k) = \beta_k; \quad k = 2, 3, \dots, n+1$$

and

$$(7) \quad P'''_{3n+1}(x_k) = \gamma_k; \quad k = 2, 3, \dots, n+1$$

provided x_k 's are the zeros of $(1-x^2)T_n(x)$ and if there exists such a polynomial then there is an infinity of them.

It seems that so far no one has obtained such a result for $(0, 2)$ -interpolation. The main object of this note is to find a set of $n+2$ points satisfying (3) for which there does not exist a unique polynomial $Q_{2n+1}(x)$ of degree $\leq 2n+1$ satisfying (4) regardless whether n is even or n is odd. More precisely we establish the following:

Theorem 2. *If n is a positive integer, $\{x_v\}_{1}^{n+2}$ are the zeros of*

$$(1-x^2)P_n^{(-\frac{1}{3}, -\frac{1}{3})}(x), \quad \text{where } P_n^{(-\frac{1}{3}, -\frac{1}{3})}(x)$$

is the Jacobi polynomial of degree n with $(\alpha, \beta) = (-\frac{1}{3}, -\frac{1}{3})$, and $\{\alpha_v\}_{1}^{n+2}$ and $\{\beta_v\}_{2}^{n+1}$ are arbitrary real numbers then there is, in general, no polynomial $Q_{2n+1}(x)$ of degree $\leq 2n+1$ such that

$$(8) \quad Q_{2n+1}(x_v) = \alpha_v; \quad v = 1, 2, \dots, n+2$$

and

$$(9) \quad Q''_{2n+1}(x_v) = \beta_v; \quad v = 2, 3, \dots, n+1,$$

and if there exists such a polynomial then there is an infinity of them.

PROOF. Our aim is here to show that when $\alpha_v=0$, $v=1, 2, \dots, n+2$ and $\beta_v=0$; $v=2, 3, \dots, n+1$ there exists a polynomial $Q_{2n+1}(x)$ of degree $\leq 2n+1$ which is not identically zero but satisfies (8) and (9). Thus invoking a well known theorem on the solution of a system of equations we prove Theorem 2.

Due to the conditions of the theorem it follows that

$$(10) \quad Q_{2n+1}(x) = (1-x^2)w_n(x)r_{n-1}(x)$$

where $w_n(x) = P_n^{(-\frac{1}{3}, -\frac{1}{3})}(x)$ and $r_{n-1}(x)$ is a polynomial of degree $\leq n-1$. Since $Q''_{2n+1}(x_v)=0$ for $v=2, 3, \dots, n+1$ we have

$$(1-x_v^2)r'_{n-1}(x_v) - \frac{4}{3}x_v r_{n-1}(x_v) = 0; \quad v = 2, 3, \dots, n+1$$

which implies that

$$(11) \quad (1-x^2)r'_{n-1}(x) - \frac{4}{3}xr_{n-1}(x) = cw_n(x)$$

where c is a non zero numerical constant. Setting

$$(12) \quad r_{n-1}(x) = \sum_{k=0}^{n-1} a_k w_k(x)$$

and using the recurrence relation ([11], formula (4.5.7), p. 72)

$$(13) \quad (1-x^2)w'_n(x) = \left(n - \frac{1}{3}\right)w_{n-1}(x) - nxw_n(x)$$

we have from (11):

$$(14) \quad \sum_{k=1}^{n-1} \left(k - \frac{1}{3}\right) a_k w_{k-1}(x) - \sum_{k=0}^{n-1} a_k \left(k + \frac{4}{3}\right) xw_k(x) = cw_n(x).$$

From (14) and formula (4.5.1) of SZEGŐ [11], p. 71 we conclude that

$$cw_n(x) = \sum_{k=1}^{n-2} \left[\frac{k(3k+2)}{6k} a_{k+1} - \frac{k(3k+1)(3k-2)}{(6k-5)(3k-1)} a_{k-1} \right] w_k(x) - \frac{(3n-2)(n-1)(3n-5)}{(6n-11)(3n-4)} a_{n-2} w_{n-1}(x) - \frac{n(3n+1)(3n-2)}{(6n-5)(3n-1)} a_{n-1} w_n(x).$$

Now equating the coefficients of $w_k(x)$ on both sides we get

$$(15) \quad \frac{n(3n+1)(3n-2)}{(6n-5)(3n-1)} a_{n-1} = -c,$$

$$(16) \quad \frac{(3n-2)(n-1)(3n-5)}{(6n-11)(3n-4)} a_{n-2} = 0$$

and

$$(17) \quad \frac{k(3k+2)}{6k} a_{k+1} - \frac{k(3k+1)(3k-2)}{(6k-5)(3k-1)} a_{k-1} = 0; \quad k = 1, 2, \dots, n-2.$$

If n is even then from the above equations (15), (16) and (17) we have $a_{n-2} = a_{n-4} = \dots = a_2 = a_0 = 0$ and $a_1, a_3, a_5, \dots, a_{n-1}$ can be determined and are not zero. Similarly, if n is odd we have from the above equations $a_1 = a_3 = a_5 = \dots = a_{n-2} = 0$ and $a_0, a_2, a_4, \dots, a_{n-1}$ are not zero and can be determined. Hence regardless whether n is even or odd, in general, there does not exist a unique polynomial $Q_{2n+1}(x)$ of degree $\leq 2n+1$ satisfying (8) and (9) and there are infinitely many if they exist.

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