

Lattice ordered rings

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It is noted in the monograph of L. FUCHS [1] that the general theory of lattice ordered rings is still not developed. It considers predominately the so-called *F*-rings which are subrings and sublattices of a complete direct sum of linear ordered rings. With regard to this Fuchs puts the following problem No 34: "When is a ring isomorphic to a lattice ordered ring?, i.e. under what condition does a given ring allow a lattice order."

The present paper gives a necessary and sufficient condition for an extension of the partial order in a ring to be a latticed order. In the particular case of the trivial order $P = \{0\}$ an answer is obtained to Fuchs' problem.

Definition. We shall say, that the partially ordered ring R is lattice ordered, if its additive group is lattice ordered, i.e. for each pair $g, g' \in R$, $g \neq g'$ there exists one $s \in R$ such that $s - g \geq 0$ $s - g' \geq 0$ and for any $t \in R$ with $t - g \geq 0$, $t - g' \geq 0$ one has $t - s \geq 0$.

If R is a directedly ordered ring, then for each $g, g' \in R$, $g \neq g'$ there exists a nonvain set of the elements s of R having the conditions $s - g \geq 0$, $s - g' \geq 0$. It is denoted by $S(g, g')$. Obviously then $H(s - g, s - g')$ is a conic semiring for every $s \in S(g, g')$.

Let R be such a ring, that for every $g, g' \in R$ there exists $S(g, g') \neq 0$. Let $\bar{R}$ be the sign for the set of all ordered pairs of different elements from R , i.e. $\bar{R} = R^2 \setminus \text{diag } R$. Then there is a function $\bar{F}$ defined on $\bar{R}$ with values in $S(g, g')$.

Theorem. *The partial order P in the ring R can be extended to a lattice order if and only if it satisfies the following condition:*

(A) For every $(g, g') \in \bar{R}$ the set $S(g, g')$ is nonvain and there exists a function $\bar{F}$, so that if $(g_1, g'_1), \dots, (g_n, g'_n) \in \bar{R}$ and $s_1, \dots, s_n$ are their corresponding elements by $\bar{F}$ and if

$$t_1 - g_1, t_1 - g'_1 \in H(P, s_1 - g_1, s_1 - g'_1) = P_1,$$

.....

$$t_n - g_n, t_n - g'_n \in H(P_{n-1}, s_n - g_n, s_n - g'_n) = P_n.$$

then the semiring

$$H(P, s_1 - g_1, s_1 - g'_1, \dots, s_n - g_n, s_n - g'_n, t_1 - s_1, \dots, t_n - s_n)$$

is a conic semiring (c.s.)

PROOF. The *necessity* is obvious. Indeed let R be a ring, which admits an order P that can be extended to a lattice order P^* . Then according to the definition there

Corollary. *A ring R admits a lattice order if and only if the following condition is satisfied:*

[illegible]
$$H(s_1 - g_1, s_1 - g'_1, \dots, s_n - g_n, s_n - g'_n, t_1 - s_1, \dots, t_n - s_n)$$

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[1] L. FUCHS, Partially ordered algebraic systems, 1963.

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