Divisibility properties of arithmetical functions

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1. Let us suppose that $g_1(n), \dots, g_k(n)$ are additive arithmetical functions having only non-negative integer values. Let $\underline{m}$ denote vectors the components m_i of which are non-negative integers. Let

$$\underline{g}(n) = \big(g_1(n), \dots, g_k(n)\big).$$

The purpose of this paper is to calculate the asymptotical density of the sequence n satisfying the relation $\underline{g}(n)=\underline{m}$, for fixed $\underline{m}$'s, and for a large class of functions.

Let

$$N(x, \underline{m}) = \sum_{\substack{n \leq x \\ \underline{g}(n)=\underline{m}}} 1. \tag{1.1}$$

Let $\mathbf{P}$ denote the set of primes, and for every feasible $\underline{m}$ let $\mathbf{P}_{\underline{m}}$ be the set of primes p satisfying the relation $\underline{g}(p)=\underline{m}$. Let $\pi_{\underline{m}}(x)$ be the number of elements in $\mathbf{P}_{\underline{m}}$ that do not exceed x. It is obvious that

$$\sum_{\underline{m}} \pi_{\underline{m}}(x) = \pi(x). \tag{1.2}$$

Furthermore we shall suppose that the inequality

$$\sum_{\underline{m}} |\pi_{\underline{m}}(x) - c_{\underline{m}}\pi(x)| \leq \frac{Kx}{(\log x)^{1+\gamma}} \tag{1.3}$$

holds, where K and γ are positive constants, $c_{\underline{m}} \geq 0$ for every $\underline{m}$.

From (1.2) we have easily that

$$\sum_{\underline{m}} c_{\underline{m}} = 1. \tag{1.4}$$

On the assumption of (1.3) we can determine the asymptotical behaviour of $N(x, \underline{m})$.

Let

$$G_{\underline{m}}(s) = \sum_{\underline{g}(n)=\underline{m}} \frac{1}{n^s}, \tag{1.5}$$

s being a complex variable. Let

$$e_{\underline{m}}(p^l) = \begin{cases} 1 & \text{when } \ \underline{g}(p^l) = \underline{m}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

We investigate the asymptotic of $G_{\underline{m}}(s)$ at $s=1$. Then, by using a tauberian theorem due to H. DELANGE [1] — which we state as Lemma 1 — we get almost immediately the asymptotic of $N(x, \underline{m})$.

2. Lemma 1. (1) *If* $a_n \geqq 0$, *b is a real number and for* $\operatorname{Re} s > b$ *we have*

$$\sum_{n=1}^{\infty} a_n \cdot n^{-s} = (s-b)^{-a} \sum_{j=0}^{m} h_j(s) \log^j \left(1/(s-b)\right) \tag{2.1}$$

where a is positive, $h_0(s), \dots, h_m(s)$ *are regular for* $\operatorname{Re} s \geqq b$ *and* $h_m(b) \neq 0$, *then*

$$\sum_{n \leqq x} a_n \sim b^{-1} h_m(b) \Gamma^{-1}(a) x^b (\log\log x)^m (\log x)^{a-1}. \tag{2.2}$$

(2) *If* $a_n \geqq 0$ *and*

$$\sum_{n=1}^{\infty} a_n \cdot n^{-s} = h_0(s)(s-1)^{-a} + \sum_{j=1}^{m} h_j(s)(s-1)^{-c_j} + h(s)$$

for $\operatorname{Re} s > 1$, *where a is a real number not equal to zero or a negative integer,* $h_0(s)$, $\dots, h_m(s)$, $h(s)$ *are regular for* $\operatorname{Re} s \geqq 1$, $h_0(1) \neq 0$ *and the constants* c_j *are real numbers not equal to zero or a negative integer all less than a, then*

$$\sum_{n \leqq x} a_n \sim h_0(1) \Gamma^{-1}(a) x (\log x)^{-1}.$$

(3) *If* $a_n \geqq 0$ *and*

$$\sum_{n=1}^{\infty} a_n \cdot n^{-s} = \sum_{j=0}^{m} h_j(s) \log^j \left(1/(s-1)\right)$$

for $\operatorname{Re} s > 1$, *where* $h_0(s), \dots, h_m(s)$ *are regular for* $\operatorname{Re} s > 1$, $m \geqq 1$ *and* $h_m(1) \neq 0$, *then*

$$\sum_{n \leqq x} a_n \sim m h_m(1) x (\log\log x)^{m-1} (\log x)^{-1}.$$

3. The case $\underline{m} = \underline{0}$.

It is obvious that

$$G_{\underline{0}}(s) = \prod_p \left(1 + \sum_{l=1}^{\infty} e_{\underline{0}}(p^l) p^{-ls}\right), \tag{3.1}$$

and the product converges absolutely in the halfplane $\operatorname{Re} s > 1$.

a. *The case* $c_{\underline{0}} > 0$. We shall prove that the function

$$u_{\underline{0}}(s) = G_{\underline{0}}(s) \cdot [\zeta(s)]^{-c_{\underline{0}}} \tag{3.2}$$

is regular in the halfplane $\operatorname{Re} s \geqq 1$, and that $u_0(1) \neq 0$.

For this, we take

$$\log u_{\underline{0}}(s) = \log G_{\underline{0}}(s) - c_{\underline{0}} \log \zeta(s) = \sum_{p \in P_{\underline{0}}} \frac{e_{\underline{0}}(p)}{p^s} - c_{\underline{0}} \sum_p 1/p^s + v_0(s),$$

where $v_0(s)$ is regular and bounded in $\operatorname{Re} s > 3/4$, say. Since

$$\sum_{p \in P_0} \frac{e_{\underline{0}}(p)}{p^s} = \int_1^\infty u^{-s}\, d\pi_{\underline{0}}(u) = s \int_1^\infty \frac{\pi_{\underline{0}}(u)}{u^{s+1}}\, du,$$

and

$$\sum_{p \in P} 1/p^s = s \int_1^\infty \frac{\pi(u)}{u^{s+1}}\, du,$$

therefore

$$\log u_{\underline{0}}(s) = s \int_1^\infty \frac{\pi_{\underline{0}}(u) - c_{\underline{0}}\pi(u)}{u^{s+1}}\, du + v_0(s). \tag{3.3}$$

So, by (1.3) the last integral is convergent for $\operatorname{Re} s \geqq 1$. Since $\zeta(s)(s-1) \to 1$ for $s \to 1$, we get

$$G_{\underline{0}}(s) = h_0(s)(s-1)^{-c_{\underline{0}}} \tag{3.4}$$

where

$$h_0(s) = \frac{G_{\underline{0}}(s)}{(s-1)^{c_{\underline{0}}}} \left(\frac{s-1}{\zeta(s)}\right)^{c_{\underline{0}}}. \tag{3.5}$$

By Lemma 1, we get

$$N(x, \underline{0}) \sim A(\underline{0})\, \Gamma(c_{\underline{0}})^{-1} x (\log x)^{c_{\underline{0}}-1} \tag{3.6}$$

where $A(\underline{0})$ being a positive constant defined by

$$A(\underline{0}) = \lim_{s \to 1+0} G_{\underline{0}}(s) \cdot (s-1)^{-c_{\underline{0}}}. \tag{3.7}$$

b. *The case* $c_{\underline{0}} = 0$.

In this case we can prove only the inequality

$$N(x, \underline{0}) \ll x \exp\left(\sum_{p<x} \frac{e_{\underline{0}}(p) - 1}{p}\right), \tag{3.8}$$

which by

$$\sum_p \frac{e_{\underline{0}}(p)}{p} < \infty$$

gives that

$$N(x, \underline{0}) \ll x/\log x. \tag{3.9}$$

(3.8) is a special case of a general well known theorem.

4. The case $\underline{m} \neq \underline{0}$.

For a general natural number K, let $G_{\underline{0}}(s|K)$ be defined as

$$G_{\underline{0}}(s|K) = \sum_{\substack{(n,K)=1 \\ g(n)=\underline{0}}} 1/n^s. \tag{4.1}$$

Then

$$G_{\underline{0}}(s|K) = \prod_{p \nmid K} \left(1 + \sum_{l=1}^\infty \frac{e_{\underline{0}}(p^l)}{p^{ls}}\right). \tag{4.2}$$

Hence

(4.3) $$G_{\underline{0}}(s|K) = A(s|K) \cdot G_{\underline{0}}(s),$$

where

(4.4) $$A(s|K) = \prod_{p/K} \left(1 + \sum_{l=1}^{\infty} \frac{e_{\underline{0}}(p^l)}{p^{ls}}\right)^{-1}.$$

Let B denote the set of those integers K, for every exact prime power divisor $(p^\alpha \| K)$ of which $\underline{g}(p^\alpha) \neq \underline{0}$.

Thus we get the relation

(4.5) $$G_{\underline{m}}(s) = \sum_{\substack{K \in B \\ \underline{g}K = \underline{m}}} \frac{1}{K^s} G_{\underline{0}}(s|K).$$

Hence, by (4.2) we get

(4.6) $$G_{\underline{m}}(s) = G_{\underline{0}}(s) \cdot H_{\underline{m}}(s),$$

where

(4.7) $$H_{\underline{m}}(s) = \sum_{\substack{K \in B \\ \underline{g}(K) = \underline{m}}} \frac{1}{K^s} A(s|K).$$

We can suppose that $\underline{g}(K) = \underline{m}$ is soluble for at least one $K \in B$, since in the opposite case $N(x, \underline{m}) = 0$, and $G_{\underline{m}}(s) \equiv 0$.

For every $K \in B$ let

(4.8) $$K = L_1 L_2 R, \quad L = L_1 L_2,$$

where L_1 contains exactly those exact prime power divisors $p^\alpha \| K$ for which $\alpha \geqq 2$ (if it is an empty set then $L_1 = 1$), L_2 contains those primes $p (\| K)$ for which $c_{\underline{g}(p)} = 0$ (in (1.3)) and R contains the other prime divisors of K. This representation of K is unique.

Taking into account that $A(s/n)$ is a multiplicative function of n, we get

(4.9) $$A(s/K) = A(s/L) \cdot A(s/R).$$

Let $\mathbf{L}, \mathbf{L}_1, \mathbf{L}_2$ be the set of the integers that occur as L, L_1, L_2, respectively. From (4.9) we have

(4.10) $$H_{\underline{m}}(s) = \sum_{L \in \mathbf{L}} \frac{A(s/L)}{L^s} B(s|L;\ \underline{m} - \underline{g}(L)),$$

where

(4.11) $$B(s/L;\ \underline{r}) = \sum \frac{1}{R^s} \cdot A(s/R),$$

he sum is extended over those R for which $R \in B$, $(R, L) = 1$ and $\underline{g}(R) = \underline{r}$.

Let D denote the set of those vectors $\underline{m}$ for which $c_{\underline{m}} > 0$. Let $R(\underline{r})$ denote the set of the solutions of the equation

(4.12) $$l_1 \cdot \underline{r}_1 + \dots + l_t \cdot \underline{r}_t = \underline{r},$$

where l_i are positive integers, $\underline{r}_i \in D$, $\underline{r}_i \neq 0$ $(i = 1, \dots, t)$.

Let $S=(\underline{r}_1, \dots, \underline{r}_t, l_1, \dots, l_t)$ denote a particular solution of (4.12). We say that an R in the right hand side of (4.11) belongs to S, when after a suitable permutation of prime-factors of $R=p_1 \dots p_s$ $(s=l_1+l_2+\dots+l_t)$ we get

$$\begin{cases} \underline{g}(p_1) = \dots = \underline{g}(p_{l_1}) = \underline{r}_1 \\ \underline{g}(p_{l_1+1} = \dots = \underline{g}(p_{l_1+l_2}) = \underline{r}_2 \\ \vdots \end{cases} \tag{4.13}$$

Then, from (4.11) we have

$$B(s/L;\, r) = \sum_{S \in \underline{R}(r)} \Delta(s/L, S), \tag{4.14}$$

where

$$\Delta(s/L, S) = \sum \frac{A(s/R)}{R^s} \tag{4.15}$$

and the sum is extended over those R for which $R \in B$, $(R, L)=1$ and $\underline{g}(R) \in S$.

Let $\mathbf{T}$ be an arbitrary set of primes. We have

$$\prod_{p \in \mathbf{T}} \left(1+\frac{zA(s/p)}{p^s}\right) = \exp\left(\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l a_l(s)\right), \tag{4.16}$$

where

$$a_l(s) = \sum_{p \in \mathbf{T}} \frac{A(s/p)^l}{p^{ls}}. \tag{4.17}$$

Let

$$H_n(s, \mathbf{T}) = \sum_{\substack{p_1 < \dots < p_n \\ p_i \in \mathbf{T}}} \frac{A(s/p_1) \dots A(s/p_n)}{(p_1 \dots p_n)^s}. \tag{4.18}$$

Taking into account that the functions in (4.16) can be expanded in Taylor series of z, comparing the coefficients we get

$$H_n(s, \mathbf{T}) = \operatorname*{coeff}_{z^n} \exp\left(\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l a_l(s)\right) = \tag{4.19}$$

$$= \operatorname*{coeff}_{z^n} \left\{\sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left(\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l a_l(s)\right)^{\nu}\right\} = \frac{1}{n!} a_1^n(s) + \sum_{j=0}^{n-1} h_j(s) \cdot a_1^j(s) + h(s)$$

where $h_j(s)$ $(j=0, \dots, n-1)$ and $h(s)$ are bounded and regular functions in the halfplane $\operatorname{Re} s > 3/4$. The bound does not depend on T.

Let now $\mathbf{T}_i$ be the set of those primes p for which $\underline{g}(p)=\underline{r}_i$, $p \nmid L$.

Then we get

$$\sum_{p \in \mathbf{T}_i} \frac{A(s/p)}{p^s} = \sum_{p \in \mathbf{T}_i} 1/p^s + v_1(s) = \sum_{p \in \mathbf{P}_{\underline{r}_i}} 1/p^s + v_1(s) - \sum_{\substack{p/L_1 \\ \underline{g}(p)=\underline{r}_i}} 1/p^s, \tag{4.20}$$

where $v_1(s)$ is bounded and regular in $\operatorname{Re} s > 3/4$.

First we observe that

$$(4.21)\qquad \sum_{p\in P_{\underline{r}_i}} 1/p^s = \int_0^\infty \frac{d\pi_{\underline{r}_i}(u)}{u^s} = c_{\underline{r}_i}\log\frac{1}{s-1} + s\int_0^\infty \frac{\pi_{\underline{r}_i}(u)-c_{\underline{r}_i}\pi(u)}{u^{s+1}}\,du$$

and that the last integral is absolutely convergent in $\operatorname{Re} s \geqq 1$ see (1.3). We estimate the last sum in (4.20). Suppose that $\operatorname{Re} s \geqq 1$. Then

$$\sum_{\substack{p/L_1\\ \underline{g}(\mathbf{p})=\underline{r}_i}} 1/p^s \leqq \sum_{p/L_1} 1/p \leqq \log \prod_{p/L_1} \frac{1}{1-\frac{1}{p}} = c(L_1).$$

Consequently the left side of (4.20) is equal to

$$(4.22)\qquad c_{\underline{r}_i}\log\frac{1}{s-1} + v_2(s) + v(s/L, \underline{r}_i)$$

where $v_2(s)$ and $v(s/L, \underline{r}_i)$ are regular and bounded in $\operatorname{Re} s \geqq 1$, $v_2(s)$ does not depend on L, and

$$|v(s/L, r_i)| \leqq c(L_1).$$

Hence, by (4.19) we get

$$(4.23)\qquad \Delta(s|\mathbf{L}, S) = B(S)\left(\log\frac{1}{s-1}\right)^{\nu} + \sum_{j=0}^{\nu-1} t_j(s, L)\left(\log\frac{1}{s-1}\right)^{j},$$

where

$$(4.24)\qquad B(S) = \prod_{i=1}^{t} \frac{c_{\underline{r}_i}^{l_i}}{l_i!}, \quad \nu = \nu(S) = l_1 + \dots + l_t,$$

and

$$|t_j(s, L)| \ll c(L_1)^{\nu} \quad \text{in} \quad \operatorname{Re} s \geqq 1.$$

For a fixed $\underline{r}$ let

$$(4.25)\qquad \mu = \mu(\underline{r}) = \max_{S} \nu(S),$$

where $\mu(\underline{r})=0$ when $R(\underline{r})$ is an empty set. The value of μ does not depend on L.

Let

$$(4.26)\qquad A_{\underline{m}} = \max_{L_1} \mu\big(\underline{r} - \underline{g}(L_1)\big).$$

Suppose that $A_{\underline{m}} > 0$. Then we get — see (4.23), (4.15), (4.14), (4.10) —

$$(4.27)\qquad H_{\underline{m}}(s) = A_{\underline{m}}(s)\left(\log\frac{1}{s-1}\right)^{A_{\underline{m}}} + \sum_{j=0}^{A_{\underline{m}}-1} B_j(s)\left(\log\frac{1}{s-1}\right)^{j},$$

where the functions $A_{\underline{m}}(s)$, $B_j(s)$ are regular in $\operatorname{Re} s \geqq 1$, and $A_{\underline{m}}(1) \neq 0$.

By (4.6) we get

$$G_{\underline{m}}(s) = (s-1)^{-c_{\underline{0}}} \sum_{j=0}^{A_{\underline{m}}} g_j(s)\cdot\left(\log\frac{1}{s-1}\right)^{j},$$

$$g_{A_{\underline{m}}}(1) \neq 0.$$

This relation holds for $A_{\underline{m}}=0$ too, since we assumed that $\underline{g}(n)=\underline{m}$ has at least one solution. Hence, by Lemma 1 we get:

$$\text{for} \quad c_{\underline{0}} > 0$$

$$N(x, \underline{m}) \sim B_{\underline{m}} x(\log x)^{-c_{\underline{0}}-1}(\log\log x)^{A_{\underline{m}}}, \tag{4.28}$$

for $c_{\underline{0}}=0$, $A_{\underline{m}}>0$,

$$N(x, \underline{m}) \sim B_m x(\log x)^{-1}(\log\log x)^{A_{\underline{m}}-1} \tag{4.29}$$

where $B_{\underline{m}}$ is a positive constant.

5. Theorem 1. *On the assumption* (1.3), *by the notation* (4.12), (4.24), (4.25), (4.26) *we get:*

(1) *If* $c_{\underline{0}}>0$ *and there exists at least one solution of* $\underline{g}(n)=\underline{m}$, *then*

$$N(x, \underline{m}) \sim B_{\underline{m}} x(\log x)^{-c_{\underline{0}}-1}(\log\log x)^{A_{\underline{m}}}.$$

(2) *If* $c_{\underline{0}}=0$ *and* $A_{\underline{m}}>0$, *then*

$$N(x, \underline{m}) \sim B_{\underline{m}} x(\log x)^{-1}(\log\log x)^{A_{\underline{m}}-1},$$

$B_{\underline{m}}$ *are suitable positive constants.*

As it is easy to see from our result follows the assertion due to W. Narkiewicz [2] concerning the divisibility properties of integer-valued multiplicative functions defined as values of a polynomial for every prime.

6. Let $f_1(n), \ldots, f_k(n)$ be multiplicative functions having positive integer values. Let $q_1, \ldots, q_k$ be arbitrary not necessarily distinct prime numbers. We define $g_i(n)$ as the greatest α for which q_i^α is a divisor of $f_i(n)$. It is clear that $g_i(n)$ are additive functions having non-negative integer values.

Assuming that the relation (1.3) holds for

$$\underline{g}(n) = \big(g_1(n), \ldots, g_k(n)\big),$$

we can use Theorem 1 to determine the asymptotic behaviour of

$$N(x, \underline{m}) = \sum_{\substack{n \leq x \\ \underline{g}(n)=\underline{m}}} 1.$$

To illustrate our theorem we investigate the divisibility properties of the Euler totient function.

Let $q_1<q_2<\ldots<q_k$ be distinct primes. Let $\underline{m}=(\alpha_1, \alpha_2, \ldots, \alpha_k)$, α_i be non-negative integers. Let $D=q_1^{\alpha_1}\ldots q_k^{\alpha_k}$. We say that D is a total divisor of N, if D/N and $q_i^{\alpha_i+1}\nmid N$ $(i=1, \ldots, k)$. We write then $D\| N$. Let $N_D(x)$ denote the number of $n\leq x$ satisfying the relation $D\| n$.

Let $\pi_{\underline{m}}(x)$ be the number of primes p not exceeding x for which $D\| p-1$. Let $Q=q_1\dots q_k$. We have

$$\pi_{\underline{m}}(x) = \sum_{\substack{p\equiv 1 \bmod D\\ \left(\frac{p-1}{D}, Q\right)=1}} 1 = \sum_{\delta/Q}\mu(\delta)\sum_{\substack{p\equiv 1 (\bmod D\delta)\\ p\leqq x}} 1 = \sum_{\delta/Q}\mu(\delta)\pi(x, \delta D, 1)$$

where in general $\pi(x, k, l)$ denotes the number of primes $p\leqq x$ in the arithmetical progression $l\,(\bmod k)$. Using the prime number theorem for arithmetical progression see e.g. K. Prachar [3] we get

$$\pi_{\underline{m}}(x) = c_{\underline{m}}\, lix + O\big(x/(\log x)^{20}\big)$$

uniformly for $D\leqq(\log x)^{10}$, where

$$c_{\underline{m}} = \sum_{\delta/Q}\frac{\mu(\delta)}{\varphi(D\delta)}. \tag{6.1}$$

Taking into account that

$$\pi_{\underline{m}}(x) \ll \frac{lix}{\varphi(D)} \quad \text{for} \quad D\leqq\sqrt{x}$$

and

$$\pi_{\underline{m}}(x) \ll \frac{x}{D} \quad \text{for} \quad D\leqq x$$

we deduce easily that (1.3) holds. Now

$$c_{\underline{0}} = \sum_{\delta/Q}\frac{\mu(\delta)}{\varphi(\delta)} = \prod_{i=1}^{k}\left(1-\frac{1}{q_i-1}\right). \tag{6.2}$$

We see that $c_{\underline{0}}>0$ if $q_1>2$ and $c_{\underline{0}}=0$ if $q_1=2$. Furthermore,

$$c_{\underline{m}} = \frac{1}{\varphi(D)}\prod_{i=1}^{k}\left(1-\frac{1}{q_i}\right). \tag{6.3}$$

Thus from theorem 1 we immediately deduce

Theorem 2. (1) *If* $q_1>2$, *then*

$$N_D(x) \sim B_D x(\log x)^{-c_{\underline{0}}-1}(\log\log x)^{\alpha_1+\dots+\alpha_k}.$$

(2) *If* $q_1=2$, $\alpha_1=0$, *then* $N_D(x)=1$ *for* $x\geqq 2$.

(3) *If* $q_1=2$, $\alpha_1>1$, *then*

$$N_D(x) \sim B_D x(\log x)^{-1}(\log\log x)^{\alpha_1-1}.$$

Proof. Assume that $q_1>2$. Then $c_{\underline{r}}>0$ for every $\underline{r}$, consequently

$$\mu(\underline{r}) = \text{ the sum of the components of } \underline{r}$$

(see (4.12), (4.25)). Since

$$A_{\underline{m}} = \max_{L_1 \in \mathbf{L}_1} \mu\big(\underline{r} - g(L_1)\big),$$

the maximum is taken for $L_1 = 1$. Thus

$$A_{\underline{m}} = \alpha_1 + \ldots + \alpha_k,$$

and we can use Theorem 1.

Assume that $q_1 = 2$. The assertion (2) is obvious, let $\alpha_1 > 2$. To prove (3) we need take into account that $c_{\underline{r}} = 0$ if and only if the first component of $\underline{r}$ is zero. We have that $A_{\underline{m}} = \alpha_1$ and so we use Theorem 1.

References

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[2] W. NARKIEWICZ, Divisibility properties of a class of multiplicative functions, *Colloquium Mathematicum*, **18** (1967), 219—232.

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