

Nagata's metric for uniformities

By S. BUZÁSI (Debrecen)

J.-I. Nagata's monograph on dimension theory contains his theorem characterizing the dimension of a metric space by property of its metric function ([1], pp. 138—149). J.-I. NAGATA remarks (p. 138) that it is an open problem to find a simpler proof of the theorem. This paper gives a much shorter proof of Nagata's theorem. The relative shortness of our proof is due to formula (2) which helps to avoid Nagata's notations like (9) on p. 142 of [1]. The paper contains also a corollary about n -dimensional uniformities which generate a given n -dimensional metric topology.¹⁾

The author wishes to express her warmest thanks to Professor Á. CSÁSZÁR who carefully read the paper and gave valuable advises.

Theorem (NAGATA [1]). *A metric space R has dimension $\leq n$ iff there exists a topology preserving metric ϱ in R such that, for every $\varepsilon > 0$ and for every point $x \in R$,*

$$(n) \quad \varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon, \quad y_i \in R, \quad i = 1, \dots, n+2$$

imply $\varrho(y_i, y_j) < \varepsilon$ for some i, j with $i \neq j$.

PROOF. Necessity. Let $\dim R \leq n$; then every open covering A of R can be refined by an open covering consisting at most of $n+1$ discrete collections (see [3], theorem 1.), so there exists an open covering $\mathcal{B}$ such that $\mathcal{B}^{***} < A$ and each $B \in \mathcal{B}^{***}$ intersects at most $n+1$ elements of A (we denote $\mathcal{B}^* = \{S(B, \mathcal{B}) \mid B \in \mathcal{B}\}$ and $\mathcal{B}^{***} = ((\mathcal{B}^*)^*)^*$).

In view of this we can construct a sequence

$$(1) \quad \mathcal{U}_1 > \mathcal{U}_2^{***} > \mathcal{U}_2 > \mathcal{U}_3^{***} > \dots$$

of open coverings of R such that

- (i) mesh $\mathcal{U}_m \rightarrow 0$ as $m \rightarrow \infty$ and
- (ii) $S^2(x, \mathcal{U}_{m+1}^*)$ intersects at most $n+1$ sets of $\mathcal{U}_m$ for every $x \in R, m = 1, 2, \dots$

We shall often make use of the fact (easy to check by induction on p) that, for $\mathcal{U}_{m_i}$ from (1), $X \subset R$ and integers $1 \leq m_2 < \dots < m_p$,

$$(2) \quad S^2(\dots S^2(X, \mathcal{U}_{m_2}), \dots, \mathcal{U}_{m_p}) \subset S^3(X, \mathcal{U}_{m_2}).$$

¹⁾ In the case of a topological space R , $\dim R$ is the covering dimension of R ; for a uniform space R , ΔdR denotes the large covering dimension of R (see [2], p. 78).

Now we define a new sequence of coverings of R : for $U \in \mathcal{U}_{m_1}$ and integers $1 \leq m_1 < m_2 < \dots < m_p$ let

$$\begin{aligned} S_{m_1}(U) &= U, \\ S_{m_1 \dots m_p}(U) &= S^2(\dots S^2(U, \mathcal{U}_{m_2}), \dots, \mathcal{U}_{m_p}), \\ \sigma_{m_1 \dots m_p} &= \{S_{m_1 \dots m_p}(U) \mid U \in \mathcal{U}_{m_1}\}. \end{aligned}$$

It follows from (2) and (1) that $S_{m_1 \dots m_p}(U) \subset S^3(U, \mathcal{U}_{m_2}) \subset S(U, \mathcal{U}_{m_1})$, so we have

$$(3) \quad \sigma_{m_1} = \mathcal{U}_{m_1} < \sigma_{m_1 \dots m_p} < \mathcal{U}_{m_1}^*.$$

It is obvious that $\sigma_{m_1 \dots m_i} < \sigma_{m_1 \dots m_i \dots m_p}$ and easy to check that

$$(4) \quad \frac{1}{2^{l_1}} + \dots + \frac{1}{2^{l_q}} < \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} \quad \text{always implies}$$

$\sigma_{l_1 \dots l_q} < \sigma_{m_1 \dots m_p}$. In fact, let

$$(5) \quad m_1 = l_1, \dots, m_{i-1} = l_{i-1}, \quad m_i < l_i$$

for some $1 \leq i \leq p, q$ and $C = S_{l_1 \dots l_{i-1}}(U)$ ($C = U$ if $i = 1$) for any $U \in \mathcal{U}_{m_1}$. It follows from (2), (1) and (5) that

$$S_{l_1 \dots l_q}(U) \subset S^3(C, \mathcal{U}_{l_i}) \subset S^2(C, \mathcal{U}_{m_i}) \in \sigma_{m_1 \dots m_i} < \sigma_{m_1 \dots m_p}.$$

Let us now define a function $\varrho(x, y)$ over $R \times R$ by

$$(6) \quad \varrho(x, y) = \inf \left\{ \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} \mid y \in S(x, \sigma_{m_1 \dots m_p}) \right\},$$

$$\varrho(x, y) = 1 \quad \text{if} \quad y \notin S(x, \sigma_{m_1 \dots m_p}) \quad \text{for every} \quad \sigma_{m_1 \dots m_p}.$$

Since

$$(7) \quad S(x, \mathcal{U}_{m+1}) \subset S_m(x) = \{y \mid \varrho(x, y) < 2^{-m}\} \subset S(x, \mathcal{U}_m),$$

we conclude that $\varrho(x, y) = 0$ iff $x = y$. From (7) and (i) it follows that $\{S_m(x) \mid m = 1, 2, \dots\}$ is, for $x \in R$, a neighbourhood basis of x . Next we prove the triangle axiom for ϱ . Suppose $1 > \varrho(x, y) = a \geq b = \varrho(z, y)$ (the case of $\varrho(x, y) = 1$ is trivial). In view of (4) and (6) for a given $\varepsilon > 0$ we can choose $\sigma_{m_1 \dots m_p}$ and $\sigma_{l_1 \dots l_q}$ such that

$$(8) \quad 2 \leq p, q; \quad l_1 < m_p, \quad x \in S(y, \sigma_{m_1 \dots m_p}), \quad z \in S(y, \sigma_{l_1 \dots l_q}),$$

$$a < \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} < a + \varepsilon, \quad b < \frac{1}{2^{l_1}} + \dots + \frac{1}{2^{l_q}} < b + \varepsilon,$$

$$(9) \quad \frac{1}{2^{l_1}} + \dots + \frac{1}{2^{l_q}} < \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}}.$$

It follows from (8) and (9) that $m_i \leq l_1 \leq m_{i+1}$ for some $1 \leq i < p$. First consider the case of $m_i < l_1 < m_{i+1}$. Let $U \in \mathcal{U}_{m_1}$, $V \in \mathcal{U}_{l_1}$,

$$(10) \quad x, y \in S_{m_1 \dots m_p}(U) = A \quad \text{and} \quad z, y \in S_{l_1 \dots l_q}(V) = B.$$

Denote

$D = S_{m_1 \dots m_i}(U)$, then $A \subset S^3(D, \mathcal{U}_{m_i+1}) \subset S(D, \mathcal{U}_{m_i+1}^*) \subset S(D, \mathcal{U}_{l_1+1}^*)$. Since $B \subset S^3(V, \mathcal{U}_{l_2}) \subset S(V, \mathcal{U}_{l_1+1}^*)$, so $y \in V$ or $y \in W \in \mathcal{U}_{l_1+1}^*$ and $W \cap V \neq \emptyset$, therefore $A \cup V \subset S(S^2(D, \mathcal{U}_{l_1+1}^*), \mathcal{U}_{l_1}) \subset S^2(D, \mathcal{U}_{l_1})$ and hence $x, z \in S_{m_1 \dots m_i l_1 \dots l_q}(U)$, i.e.

$$\varrho(x, z) \leq \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_i}} + \frac{1}{2^{l_1}} + \dots + \frac{1}{2^{l_q}} < a + b + 2\varepsilon.$$

Now let $m_i = l_1$ and $1 \leq m_1, \dots, m_i$ be successing integers. If $m_1 = 1$ we have $\varrho(x, z) \leq 1 = \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_i}} + \frac{1}{2^{l_1}} < a + b + 2\varepsilon$. In the case of $1 < m_1$ we conclude from (8) and (3) that $x \in S(y, \mathcal{U}_{m_1}^*)$, $z \in S(y, \mathcal{U}_{l_1}^*)$ so $x \in S(z, \mathcal{U}_{m_1}^*) \subset S(z, \mathcal{U}_{m_1-1}) = S(z, \sigma_{m_1-1})$. Hence

$$\varrho(x, z) \leq \frac{1}{2^{m_1-1}} = \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_i}} + \frac{1}{2^{l_1}} < a + b + 2\varepsilon.$$

It remains to consider the case of $m_i = l_1$ while $m_s, \dots, m_i$ are successing integers but $m_{s-1} + 1 < m_s$ for some $2 \leq s \leq i$. Using the notations of (10) and $F = S_{m_1 \dots m_{s-1}}(U)$ we conclude that $A \subset S^3(F, \mathcal{U}_{m_s}) \subset S(F, \mathcal{U}_{m_{s-1}})$ and $B \subset S(V, \mathcal{U}_{l_1}) \in \mathcal{U}_{l_1}^* = \mathcal{U}_{m_i}^* < \mathcal{U}_{m_s-1}$, hence $A \cup B \subset S^2(F, \mathcal{U}_{m_{s-1}}) = S_{m_1 \dots m_{s-1}, m_s-1}(U)$ and therefore

$$\varrho(x, z) \leq \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_{s-1}}} + \frac{1}{2^{m_s-1}} = \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_i}} + \frac{1}{2^{l_1}} < a + b + 2\varepsilon.$$

It is clear now that ϱ is a topology preserving metric in R . Let us prove that ϱ is of the property (n). We can choose x_i ($i=1, 2, \dots, n+2$) and $1 \leq m_1 < \dots < m_p$ such that $x_i \in S(y_i, \sigma_{m_1 \dots m_p})$ and $\varrho(x, x_i) < \frac{\varepsilon}{2}$, $\varrho(x_i, y_i) < \varepsilon$, $2\varrho(x, x_i) < \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} < \varepsilon$. Then for some $U_i \in \mathcal{U}_{m_1}$ we have $x_i, y_i \in S_{m_1 \dots m_p}(U_i) \subset S(U_i, \mathcal{U}_{m_1+1}^*)$ and $x_i \in S(x, \sigma_{m_1+1, \dots, m_p+1}) \subset S(x, \mathcal{U}_{m_1+1}^*)$ so $S^2(x, \mathcal{U}_{m_1+1}^*) \cap U_i \neq \emptyset$ for $i=1, 2, \dots, n+2$. In view of (ii), for some $i \neq j$, we have $U_i = U_j = W$ so $y_i, y_j \in S_{m_1 \dots m_p}(W)$, that is $\varrho(y_i, y_j) \leq \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} < \varepsilon$.

Sufficiency. Let ϱ be a metric in R of the property (n). For a given $\varepsilon > 0$ let M be a maximal subset of R such that

$$(11) \quad \varrho(x, y) \geq \varepsilon \quad \text{for every } x, y \in M, \quad x \neq y.$$

Now put $\mathcal{A}_\varepsilon = \{\bigcup \{S_{\varepsilon/2}(x) | x \in S_\varepsilon(y)\} | y \in M\}$. If for some $z \in R$, $\text{ord } \mathcal{A}_\varepsilon(z) > n+1$, then there exist distinct $y_i \in M$ ($i=1, \dots, n+2$) such that $\varrho(S_{\varepsilon/2}(z), y_i) < \varepsilon$ and hence by (n), $\varrho(y_i, y_j) < \varepsilon$ for some $i \neq j$ which contradicts (11). Therefore $\text{ord } \mathcal{A}_\varepsilon \leq n+1$ for every $\varepsilon > 0$, and so $\Delta d(R, \varrho) \leq n$. Since $\dim R$ is the minimum of $\Delta d(R, \varrho)$ for all topology preserving metrics ϱ (see [2], theorem 15. on p. 153), we conclude that $\dim R \leq n$.

Corollary. Every topology preserving metrizable uniformity on a topological space R of $\dim R \leq n$ can be refined by a topology preserving metric uniformity μ such that $\Delta d(R, \mu) \leq n$.

PROOF. The metric ϱ of property (n) gives a uniformity on R with $\Delta dR \cong n$ and finer than the uniformity induced by the original metric d in R : in view of (i) for every $\varepsilon > 0$ there exists $\mathcal{U}_m$ in (1) such that $\text{mesh } \mathcal{U}_m < \varepsilon$ and therefore $\varrho(x, y) < 2^{-m}$, $x, y \in R$ implies $x \in S(y, \mathcal{U}_m)$, i.e. $d(x, y) < \varepsilon$, which completes the proof of the corollary.

Let us note that our corollary gives more information than the theorem 15. ([2], p. 153) about n -dimensional compatible metrizable uniformities on a metric space R of $\dim R \cong n$. According to the theorem 15. all compatible metric uniformities on R are refined by the same, at most n -dimensional uniformity on R , namely by the fine uniformity of R , which however is not metrizable in general.

References

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(Received December 20, 1975.)