

On differentiable iteration groups

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In the present paper we shall give some properties of differentiable and convex iteration groups.

Let the function f fulfil the following hypothesis:

(A) f is defined, continuous and strictly increasing in the interval

$$J = [0, a] \quad \text{and} \quad 0 < f(x) < x \quad \text{for} \quad x \in (0, a].$$

Definition 1. (cf. [4], [8], [10]). A one-parameter family of functions $\{f^t, t \in \mathbb{R}\}$ is called to be an iteration group of the function f with respect to zero, provided the following conditions are fulfilled:

1° for every $t \in (-\infty, \infty)$ the function f^t is defined in an interval $[0, \delta_t]$ where $\delta_t > 0$ and $\delta_t = a$, for $t \geq 0$,

2° for every pair $s, t \in (-\infty, \infty)$

$$(1) \quad f^s(f^t(x)) = f^{t+s}(x)$$

holds for every x for which both sides are meaningful,

$$3^\circ \quad f^1(x) = f(x) \quad \text{for} \quad x \in [0, a].$$

Remark 1. (cf. [10], [6]). If $\{f^t, t \in (-\infty, \infty)\}$ is an iteration group of the function f with respect to zero and f fulfil assumption (A) and if for an $x \in (0, a]$ the function $f^t(x)$ is continuous in $[0, \infty)$, then the function $h(t) \stackrel{\text{df}}{=} f^t(a)$, for $t \in [0, \infty)$ is continuous, strictly decreasing in $[0, \infty)$, $h[0, \infty) = (0, a]$ and

$$(2) \quad f^t(x) = \begin{cases} h(t + h^{-1}(x)), & x \in (0, a], \quad t \geq -h^{-1}(x) \\ 0, & x = 0. \end{cases}$$

The above iteration group as a function of two variables is defined in the set

$$(3) \quad D = [0, a] \times [0, \infty) \cup \{(x, t): t < 0, 0 \leq x \leq h(-t)\}$$

and it is continuous in D .

§ 1. We shall deal with the differentiable iteration groups. Let us start with two definitions:

Definition 2. An iteration group of the function f with respect to zero $\{f^t, t \in (-\infty, \infty)\}$ is called to be of the class C_0^1 if the function $F(x, t) = f^t(x)$ has continuous partial derivatives F'_x, F'_t except on the line $x=0$.

Definition 3. An iteration group $\{f^t, t \in (-\infty, \infty)\}$ of the function f with respect to zero is called to be of the class C_*^1 if it is of class C_0^1 and $f^t(x)$ has the partial derivatives on the line $x=0$ being continuous with respect to each variable.

In all theorems the following general hypothesis is assumed: (H) $\{f^t, t \in (-\infty, \infty)\}$ is an iteration group of the function f with respect to zero and f satisfies hypothesis (A).

Throughout this note h denotes the function defined by the formula

$$(4) \quad h(t) \stackrel{\text{df}}{=} f(a), \quad t \geq 0.$$

Theorem 1. The following equivalences hold:

(i) For every $x \in (0, a]$ there exists the limit $\lim_{t \rightarrow 0} f^t(x) = x$ iff h is continuous in $[0, \infty]$

(ii) $f^t(x)$ is differentiable with respect to t at zero for every $x \in (0, a]$ i. e. there exists the limit

$$(5) \quad \lim_{t \rightarrow 0} \frac{f^t(x) - x}{t} = \left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} \stackrel{\text{df}}{=} g(x), \quad x \in [0, a]$$

iff h is differentiable in $[0, \infty)$. Then $g(x) \leq 0$ for $x \in (0, a]$ and $g(0) = 0$.

PROOF. (i) Let h be continuous. Then, on account of Remark 1, there exists an $s \geq 0$ such that $x = f^s(a)$. From the definition of h we have $h(t+s) = f^{t+s}(a) = f^t(f^s(a)) = f^t(x)$ where $|t| < \delta$ for an $\delta > 0$. Hence $\lim_{t \rightarrow 0} f^t(x) = h(s) = x$. If $x = 0$, then $f^t(0) = 0$.

Conversely, if $\lim_{t \rightarrow 0} f^t(x) = x$, for $x \in (0, a]$, then $\lim_{t \rightarrow 0} h(t+s) = \lim_{t \rightarrow 0} f^t(f^s(a)) = f^s(a) = h(s)$, for $s \geq 0$.

(ii) If there exists the derivative (5) in J , then $\lim_{t \rightarrow 0} f^t(x) = x$ in J , so h is continuous. From Remark 1 it follows that $f^t(x) = h(t + h^{-1}(x))$. Hence we have

$$(6) \quad \lim_{t \rightarrow 0} \frac{f^t(x) - x}{t} = \lim_{t \rightarrow 0} \frac{h(t + h^{-1}(x)) - h(h^{-1}(x))}{t}.$$

The existence of one of these limits implies the existence of the other one. Hence h is differentiable in $[0, \infty)$, since $h^{-1}([0, a]) = [0, \infty)$.

Conversely, if h is differentiable, then $f^t(x) = h(t + h^{-1}(x))$ whence, by relation (6) f^t is differentiable at 0 with respect to t then, by Remark 1, we have $f^t(x) < x$ for $x \in (0, a)$, so $g(x) \leq 0$ in J .

Corollary 1. If there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J , then

$$(7) \quad g(h(t)) = h'(t), \quad t \in [0, \infty).$$

PROOF. From Theorem 1, h is differentiable. Moreover we have $g(h(t)) = \lim_{s \rightarrow 0} \frac{f^s(h(t)) - h(t)}{s} = \lim_{s \rightarrow 0} \frac{f^{s+t}(a) - f^t(a)}{s} = \lim_{s \rightarrow 0} \frac{h(t+s) - h(t)}{s} = h'(t)$.

Theorem 2. *The following equivalences hold:*

(i) *There exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ being continuous in J iff $h \in C^1[[0, \infty))$ and $\lim_{t \rightarrow \infty} h'(t) = 0$.*

(ii) *There exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ being continuous in J and differentiable at zero iff $h \in C^1[[0, \infty))$ and there exists the finite limit $\lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} = \gamma$ (then $\gamma = g'(0)$).*

PROOF. Equivalence (i) follows immediately from Corollary 1 and Theorem 1 (ii). Equivalence (ii) follows from (i) and from the relations

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x)}{x} = \lim_{t \rightarrow \infty} \frac{g(h(t))}{h(t)} = \lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)},$$

since the existence of one of these limits implies the existence of the other ones.

Theorem 3. *Suppose that there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J . Then for every $x \in [0, a)$ $f^t(x)$ is differentiable with respect to t in $[-h^{-1}(x), \infty)$ and*

$$(8) \quad \left. \frac{\partial f^t(x)}{\partial t} \right|_{t=t} \stackrel{\text{df}}{=} \left. \frac{\partial f^s(x)}{\partial s} \right|_{s=t} = g(f^t(x)).$$

If $f^t(x)$ is differentiable at x_0 for a t , then

$$(9) \quad g(f^t(x_0)) = \left. \frac{\partial f^t(x)}{\partial x} \right|_{x=x_0} g(x_0).$$

In particular, if f is differentiable in J , then

$$(10) \quad g(f(x)) = f'(x)g(x), \quad \text{for } x \in J.$$

PROOF. (See also [2], [3].) By the definition of g we have

$$g(f^t(x)) = \lim_{s \rightarrow 0} \frac{f^s(f^t(x)) - f^t(x)}{s} = \lim_{s \rightarrow 0} \frac{f^{s+t}(x) - f^t(x)}{s} = \left. \frac{\partial f^s(x)}{\partial s} \right|_{s=t}.$$

If $f^t(x)$ is differentiable with respect to x at x_0 for a t , then in view of (5)

$$\begin{aligned} g(f^t(x_0)) &= \lim_{s \rightarrow 0} \frac{f^s(f^t(x_0)) - f^t(x_0)}{s} = \\ &= \lim_{s \rightarrow 0} \frac{f^t(f^s(x_0)) - f^t(x_0)}{f^s(x_0) - x_0} \frac{f^s(x_0) - x_0}{s} = \left. \frac{f^t(x)}{x} \right|_{x=x_0} g(x_0), \end{aligned}$$

since $\lim_{s \rightarrow 0} f^s(x_0) = x_0$.

Corollary 2. Suppose that there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J .

Then:

- (i) $g(x_0) \neq 0$ for an $x_0 \in (0, a]$ iff for any $s \geq -h^{-1}(x_0)$ there exists the derivative $\left. \frac{\partial f^s(x)}{\partial x} \right|_{x=x_0}$,
- (ii) g is continuous in J and $g(x) < 0$ for $x \in (0, a]$ iff $\{f^t, t \in (-\infty, \infty)\}$ is of class C_0^1 .

PROOF. (i). If $g(x_0) \neq 0$ and $x_0 \neq 0$, then in view of (7) $h'(h^{-1}(x_0)) \neq 0$, so the formula $f^t(x) = h(t + h^{-1}(x))$ implies that for $t \geq -h^{-1}(x_0)$, $f^t(x)$ is differentiable with respect to x at x_0 .

Conversely, let $f^t(x)$ be differentiable at x_0 for $t \geq -h^{-1}(x_0)$ and $g(x_0) = 0$. Then, on account of (9), $g(f^t(x_0)) = 0$ for $t \geq -h^{-1}(x_0)$. Since $f^t(x_0) = h(t + h^{-1}(x_0))$ and $h[[0, \infty)] = (0, a]$, we get $g(x) \equiv 0$ in J , but this is impossible.

Equivalence (ii) follows immediately from (i) and Theorem 3.

Remark 2. If there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J and $g(x_0) \neq 0$, then there exists the derivative $f'(x_0)$. However, the converse theorem is not true. There exists an iteration group of the function $f(x) = sx$ for $x \in [0, 1]$, where $0 < s < 1$ such that there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J and $g(x_0) = 0$ for an $x_0 > 0$.

In fact, there exists a function h strictly decreasing, positive and differentiable in $[0, \infty)$ satisfying the equation

$$h(t+1) = sh(t), \quad \text{for } t \in [0, \infty)$$

and such that $h(0) = 1$ and $h'(0) = 0$ (see [4]).

The function h defines an iteration group of f by formula (2). This group has the property that there exists $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ and $g(h(n)) = 0$ for all integers $n \geq 0$.

Theorem 4. Suppose that there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ being continuous in J , $g(x) < 0$ for $x \in (0, a]$ and g is differentiable at zero¹⁾. Then $f^{s'}(0) = \left. \frac{\partial f^s(x)}{\partial x} \right|_{x=0} = \exp s \cdot g'(0) = [f'(0)]^s$. If moreover $-\infty < g'(0) \neq 0$, then the iteration group $\{f^t, t \in (-\infty, \infty)\}$ is of class C_*^1 .

PROOF. We may assume that $s > 0$, since $f^{-s} = (f^s)^{-1}$ in a neighbourhood of zero, and $f^0(x) = x$ in J . From Theorem 2 (i) and Corollary 1 it follows that

¹⁾ We admit $g'(0) = -\infty$ ($\exp(-\infty) \stackrel{\text{def}}{=} 0$).

$\ln h \in C^1[[0, \infty)]$, where $h(t) = f^t(a)$. Then in view of Lagrange theorem, for any $t > 0$ there exists an $\Theta_t \in (0, s)$ such that

$$\frac{\ln h(t+s) - \ln h(t)}{s} = \frac{h'(t+\Theta_t)}{h(t+\Theta_t)}.$$

Furthermore, according to Theorem 2 (ii) we infer that there exists the limit

$$(11) \quad \lim_{t \rightarrow \infty} \frac{h'(t)}{h(t)} = g'(0),$$

so

$$\lim_{t \rightarrow \infty} \frac{h'(t+\Theta_t)}{h(t+\Theta_t)} = g'(0).$$

Consequently

$$(12) \quad \lim_{t \rightarrow \infty} \ln \frac{h(t+s)}{h(t)} = \lim_{t \rightarrow \infty} (\ln h(t+s) - \ln h(t)) = sg'(0),$$

whence $f^{s'}(0) = \exp sg'(0)$. For $s=1$ we have $f'(0) = \exp g'(0)$, then $f^{s'}(0) = f'(0)^s$.

Let $-\infty < g'(0) \neq 0$. By Corollary 2 (ii), $\{f^t : t \in (-\infty, \infty)\}$ is of class C_0^1 . Moreover, we have shown above that $f^t(x)$ is differentiable with respect to t on the line $x=0$. By (11) and (12), we have the relation

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\partial f^s(x)}{\partial x} &\stackrel{\text{df}}{=} \lim_{x \rightarrow 0} f^{s'}(x) = \lim_{x \rightarrow 0} \frac{h'(s+h^{-1}(x))}{h'(h^{-1}(x))} = \lim_{t \rightarrow \infty} \frac{h'(s+t)}{h'(t)} = \\ &= \lim_{t \rightarrow \infty} \frac{h'(t+s)}{h(t+s)} \frac{h(t+s)}{h(t)} \frac{h(t)}{h'(t)} = \lim_{t \rightarrow \infty} \frac{h'(t+s)}{h(t+s)} \lim_{t \rightarrow \infty} \frac{h(t+s)}{h(t)} \lim_{t \rightarrow \infty} \frac{h(t)}{h'(t)} = \\ &= g'(0) f^{s'}(0) \frac{1}{g'(0)} = f^{s'}(0) = f'(0)^s = \left. \frac{\partial f^s(x)}{\partial x} \right|_{x=0}. \end{aligned}$$

The above limits exist, since $g'(0) \neq 0$.

Moreover, $\lim_{x \rightarrow 0} \frac{\partial f^s(x)}{\partial s} = \lim_{x \rightarrow 0} g(f^s(x)) = 0$, and $\left. \frac{\partial f^s(x)}{\partial s} \right|_{x=0} \equiv 0$. Hence $\{f^t, t \in (-\infty, \infty)\}$ is of class C_*^1 .

Theorem 5. (Cf. [1], [2].) *The function h defined by (4) satisfies the equation*

$$f(h(t)) = h(t+1), \quad \text{for } t \in [0, \infty).$$

If h is continuous, then there exists the inverse function $\alpha(x) = h^{-1}(x)$ defined in $(0, a]$ and satisfies the Abel equation

$$\alpha(f(x)) = \alpha(x) + 1 \quad \text{for } x \in (0, a].$$

If there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x) < 0$ in $(0, a]$, then α is differentiable, $\alpha'(x) \neq 0$ in $(0, a]$ and

$$g(x) = 1/\alpha'(x), \quad \text{for } x \in (0, a].$$

Now, our assertion follows directly from Remark 1 and Corollary 1.

Theorem 6. Let g be defined and continuous in $[0, a]$, $g(x) < 0$, for $x \in (0, a]$ and let the integral

$$(13) \quad \int_0^a \frac{du}{g(u)}$$

diverge. Then there exists exactly one iteration group $\{f^t, t \in (-\infty, \infty)\}$ of class C_0^1 of a function f satisfying hypothesis (A) such that

$$(14) \quad \left. \frac{\partial f^s(x)}{\partial s} \right|_{s=0} = g(x) \quad \text{for } x \in [0, a].$$

Conversely, if $\{f^t, t \in (-\infty, \infty)\}$ is an iteration group of class C_0^1 of a function f satisfying (A), then integral (13) diverges, where g is given by (14).

PROOF. It is easy to verify, that the function $f^t(x)$ defined by formula (2) where h is a solution of a differential equation (7) such that $h(0)=a$, is an iteration group satisfying the conditions of our theorem. The uniqueness follows from Remark 1 and Corollary 1, since equation (7) has exactly one solution in $[0, \infty)$ such that $h(0)=a$.

Conversely, if $\{f^t, t \in (-\infty, \infty)\}$ is an iteration group of class C_0^1 of a function f satisfying hypothesis (A) then by Corollaries 2 (ii) and 1, $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x) < 0$, for $x \in (0, a]$ and

$$(15) \quad h^{-1}(x) = - \int_x^a \frac{du}{g(u)} \quad \text{for } x \in (0, a],$$

where $h(t)=f^t(a)$. Hence integral (13) diverges in virtue of the fact that $\lim_{x \rightarrow 0} h^{-1}(x) = \infty$.

Remark 3. (Cf. [2], [3].) If $\{f^t, t \in (-\infty, \infty)\}$ is of class C_0^1 , then

$$\frac{\partial f^t(x)}{\partial t} = \frac{\partial f^t(x)}{\partial x} g(x)$$

in $\text{Int } D$, where D is given by (3).

This equation follows directly from Theorem 3.

We have also the converse

Theorem 7. If g is continuous in J , $g(x) < 0$, for $x \in (0, a]$, $g(0)=0$ and integral (13) diverges, then there exists exactly one function F continuous in the set D (given by (3) where h^{-1} is defined by (15)) and of class C^1 in $\text{Int } D$ such that

$$(16) \quad \frac{\partial F(x, t)}{\partial t} = \frac{\partial F(x, t)}{\partial x} g(x), \quad (x, t) \in \text{Int } D$$

and $F(x, 0)=x$ for $x \in [0, a]$. This solution F is an iteration group of class C_0^1 such that $\left. \frac{\partial F(x, t)}{\partial t} \right|_{t=0} = g(x)$ in J and $F(x, 1)=f(x)$ fulfils hypothesis (A).

PROOF. From Theorem 6 and Remark 3 it follows that there exists an iteration group $\{f^t, t \in (-\infty, \infty)\}$ fulfilling hypothesis (H) of class C_0^1 defined in D and satisfying equation (16).

For every point $(x_0, t_0) \in \text{Int } D$ there exists exactly one solution z of the equation

$$(17) \quad z'(t) = -g(z(t))$$

defined in the interval $(-\infty, t_0]$, such that $z(t_0) = x_0$. This solution may be uniquely extended to a point $(\bar{t}_0, a) \in \text{Fr } D$, where $\bar{t}_0 \geq 0$ and $\bar{t}_0 \geq t_0$. From the definition of the set D it follows that $(z(t), t) \in D$ for $t \leq \bar{t}_0$. If the functions F_1 and F_2 of class C^1 in $\text{Int } D$ satisfy equation (16) and $F_i(x, 0) = x$ for $i=1, 2$ in $[0, a]$, then $F = F_1 - F_2$ also satisfies equation (16) and $F(x, 0) = 0$ in $[0, a]$. Hence by (17) we have that $F(z(t), t) = 0$ for $t \leq \bar{t}_0$, so $F(x_0, t_0) = 0$ and consequently $F \equiv 0$ in D . Thus it follows the uniqueness of solutions of equation (16) in D .

Theorem 8. Let $f^t(x_0)$ be continuous for an $x_0 \in (0, a]$ in $[0, \infty)$. If one of the functions

$$g_1(x) = \limsup_{t \rightarrow 0^+} \frac{f^t(x) - x}{t}, \quad g_2(x) = \limsup_{t \rightarrow 0^-} \frac{f^t(x) - x}{t},$$

$$g_3(x) = \liminf_{t \rightarrow 0^+} \frac{f^t(x) - x}{t}, \quad g_4(x) = \liminf_{t \rightarrow 0^-} \frac{f^t(x) - x}{t}$$

is continuous in $[0, a]$, then there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ continuous in J .

PROOF. Let g_1 be continuous in $[0, a]$. In view of Remark 1, the continuity of $f^t(x_0)$ in $[0, \infty)$ for an $x_0 \in (0, a]$ implies that $f^t(x)$ is given by formula (2). Then

$$g_1(x) = \limsup_{t \rightarrow 0^+} \frac{f^t(x) - x}{t} = \limsup_{t \rightarrow 0^+} \frac{h(t + h^{-1}(x)) - h(h^{-1}(x))}{t} = \bar{D}^+ h(h^{-1}(x)),$$

where $\bar{D}^+ h$ denotes the Dini derivative of h . Hence we have the relation

$$g_1(h(t)) = \bar{D}^+ h(t) \quad \text{for } t \geq 0.$$

Then $\bar{D}^+ h$ is continuous in $[0, \infty)$. Further, on account of Lebesgue theorem (see [5] p. 184), h is absolutely continuous, whence $\bar{D}^+ h = h'$ a.e. in $[0, \infty)$. Moreover, we have

$$h(s) = \int_0^s h'(t) dt = \int_0^s \bar{D}^+ h(t) dt.$$

Hence, by the continuity of $\bar{D}^+ h$ in $[0, \infty)$, h is of class C^1 in $[0, \infty)$. Then, by Theorem 1 (ii), there exists $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ and $g = g_1$.

§ 2. In the present section we are concerned with the convex iteration groups.

Definition 4. (See [8]) An iteration group $\{f^t, t \in (-\infty, \infty)\}$ with respect to zero is called to be convex (concave) if for any $t \geq 0$ the functions are convex (concave).

Now we give a short proof of a generalization of A. Smajdor's result (see [9]).

Theorem 9. Suppose that there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J . Then $\{f^t, t \in (-\infty, \infty)\}$ is a convex (concave) iteration group iff g is convex (concave) in J .

PROOF. If $\{f^t, t \in (-\infty, \infty)\}$ is a convex iteration group and $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J , then

$$g(x) = \lim_{n \rightarrow \infty} (f^{1/n}(x) - x)n, \quad x \in J.$$

Hence, g is convex as a limit of convex functions.

Conversely, let g be convex. Then the right-hand side derivative g'_+ is increasing in J , as well as g is continuous in J and $g(x) < 0$ for $x \in (0, a]$. Hence by Theorem 3 and Corollary 2 we have

$$(18) \quad \frac{\partial f^s(x)}{\partial x} = \frac{g(f^s(x))}{g(x)} \stackrel{\text{df}}{=} G_s(x), \quad \text{for } x \in (0, a].$$

We are going to show that G_s is increasing for any $s \geq 0$. Let $s > 0$. G_s is continuous in $(0, a]$ and right-hand side differentiable in $(0, a)$. Applying formula (18) we get

$$\begin{aligned} G'_{s+}(x) &= \frac{\frac{\partial_+ g(f^s(x))}{\partial x} g(x) - g(f^s(x))g'_+(x)}{g(x)^2} = \\ &= \frac{g'_+(f^s(x)) \frac{g(f^s(x))}{g(x)} g(x) - g(f^s(x))g'_+(x)}{g(x)^2} = \frac{g(f^s(x))[g'_+(f^s(x)) - g'_+(x)]}{g(x)^2}. \end{aligned}$$

Since $g(x) < 0$, $f^s(x) < x$ for $x \in (0, a]$ and g'_+ is increasing, we get $G'_{s+}(x) \geq 0$ in $(0, a)$. Hence G_s is increasing in $(0, a)$ in virtue of Zygmund's lemma (see [5] p. 182). Since $G_s = f^s$, the function f^s is convex in $(0, a)$. Moreover f^s is continuous at 0 and a , whence f^s is convex in J .

The proof for the concave iteration group is the same.

Theorem 10. Suppose that there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J . If $f^t(x_0)$ is convex as a function of t for an $x_0 \in (0, a]$ then g is decreasing in J . If g is decreasing in J then for every $x_0 \in (0, a]$ $f^t(x_0)$ is convex.

PROOF. From Theorem 3 (formula (8)) it follows that $f^t(x_0)$ is convex with respect to t iff $g(f^t(x_0))$ is increasing with respect to t . This condition holds iff g is decreasing, since $f^t(x_0)$ is decreasing and $f^t(x_0)$ maps the interval $[-h^{-1}(x_0), \infty)$ onto $(0, a]$, where $h(t) = f^t(a)$.

Theorem 11. There does not exist an iteration group $\{f^t, t \in (-\infty, \infty)\}$ with respect to zero of the function f fulfilling assumption (A) such that there exists the

derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ being differentiable at $(0, a]$ and $f^t(x)$ as a function of two variables is convex in $(0, a] \times [0, \infty)$.

PROOF. Suppose that $F(x, t) \stackrel{\text{df}}{=} f^t(x)$ is convex in $[0, a] \times [0, \infty)$ where $\{f^t, t \in (-\infty, \infty)\}$ satisfies assumption of our theorem. F is convex with respect to each variable. On account of Theorem 9 and 10 g is convex and decreasing in $(0, a]$ as well as $g(x) < 0$ for $x \in (0, a]$. Moreover g is of class C^1 in $(0, a]$, since g is differentiable and convex in $(0, a]$. Therefore, Theorem 3 implies that there exist the continuous derivatives

$$\frac{\partial F(x, t)}{\partial t} = g(F(x, t)) \quad \text{and} \quad \frac{\partial F(x, t)}{\partial x} = \frac{g(F(x, t))}{g(x)},$$

whence it follows that F is of class C^2 in $(0, a] \times (0, \infty)$.

Applying the criterion of convexity of functions of two variables (see [7] th. 4.5) we get that the matrix

$$\begin{vmatrix} \frac{\partial^2 F(x, t)}{\partial t^2} & \frac{\partial^2 F(x, t)}{\partial t \partial x} \\ \frac{\partial^2 F(x, t)}{\partial x \partial t} & \frac{\partial^2 F(x, t)}{\partial x^2} \end{vmatrix}$$

is nonnegatively defined for $x \in (0, a]$ and $t \in (0, \infty)$.

For our function F this is impossible. In fact, using the formulas (8) and (9) we get

$$\begin{aligned} & \begin{vmatrix} \frac{\partial^2 F(x, t)}{\partial t^2} & \frac{\partial^2 F(x, t)}{\partial t \partial x} \\ \frac{\partial^2 F(x, t)}{\partial x \partial t} & \frac{\partial^2 F(x, t)}{\partial x^2} \end{vmatrix} = \\ &= \begin{vmatrix} g'(f^t(x))g(f^t(x)) & \frac{g'(f^t(x))g(f^t(x))}{g(x)} \\ \frac{g'(f^t(x))g(f^t(x))}{g(x)} & \frac{g'(f^t(x))g(f^t(x)) - g(f^t(x))g'(x)}{g(x)^2} \end{vmatrix} = \\ &= \frac{-g'(f^t(x))g'(x)g(f^t(x))^2}{g(x)^2} \leq 0, \end{aligned}$$

since $g'(x) \leq 0$ in J as a derivative of decreasing function.

Remark 4. If $\{f^t, t \in (-\infty, \infty)\}$ is a convex (concave) iteration group and there exists the derivative $\left. \frac{\partial f^t(x)}{\partial t} \right|_{t=0} = g(x)$ in J such that $g'(0) \neq 0$, then $\{f^t, t \in (-\infty, \infty)\}$ is of class C_*^1 .

In fact, from Theorem 9 it follows that g is convex (concave), whence g is continuous, $g(x) < 0$ for $x \in (0, a)$ and there exists the derivative $g'(0) \neq 0$. Now, our assertion results from Theorem 4.

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