

On the means of an entire function of several complex variables represented by multiple Dirichlet series

By VINITA VIJAI (Lucknow)

1. Let

$$(1.1) \quad f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(s_1 \lambda_m + s_2 \mu_n)$$

($s_j = \sigma_j + it_j$, $j = 1, 2$), where $a_{m,n} \in C$, the field of complex numbers, and λ_m 's, μ_n 's are real; $0 < \lambda_1 < \lambda_2 < \dots < \lambda_m \rightarrow \infty$; $0 < \mu_1 < \mu_2 < \dots < \mu_n \rightarrow \infty$.

It has been proved [1] that if

$$(1.2) \quad \lim_{m \rightarrow \infty} \frac{\log m}{\lambda_m} = 0, \quad \lim_{n \rightarrow \infty} \frac{\log n}{\mu_n} = 0,$$

then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence. Also, SARKAR [2, pp.99] has shown that the necessary and sufficient condition for the series (1.1) satisfying (1.2) to be entire is that

$$(1.3) \quad \lim_{(m,n) \rightarrow \infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = -\infty.$$

Let the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3) be denoted by F . Then $f \in F$ denotes an entire function over C^2 . The results can be extended to several complex variables.

Corresponding to an $f \in F$, the maximum modulus $M = M_f$ and the maximum term $\mu = \mu_f$ on R^2 are defined [2, pp.100] as

$$M(\sigma) = M_f(\sigma_1, \sigma_2) = \max \{ |f(s_1, s_2)|, s_1, s_2 \in C, \operatorname{Res}_1 = \sigma_1, \operatorname{Res}_2 = \sigma_2 \}$$

$$\mu(\sigma) = \mu_f(\sigma_1, \sigma_2) = \max_{(m,n) \in N^2} \{ |a_{m,n}| \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) \},$$

where N is the set of natural numbers. The following two lemmas are due to SARKAR [2, pp.101].

Lemma A. *Let $f \in F$ be of finite order. Then $\rho = (\rho_1, \rho_2) \gg (0, 0)$ is an order point of f iff*

$$\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log \log M(\sigma)}{\log(e^{\sigma_1 \rho_1} + e^{\sigma_2 \rho_2})} \right] = 1, \quad \sigma \in R^2.$$

Lemma B. *Let $f \in F$ be of finite order. Then $\tau = (\tau_1, \tau_2) \gg (0, 0)$ is a type point of f iff*

$$\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log M(\sigma)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right] = 1, \quad \sigma \in R^2.$$

We define

$$(1.4) \quad I_p(\sigma_1, \sigma_2) = \lim_{(T_1, T_2) \rightarrow \infty} \frac{1}{4T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2$$

$$(1.5) \quad m_{p,q}(\sigma_1, \sigma_2) = \frac{4}{e^{q\sigma_1} e^{q\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} I_p(x_1, x_2) e^{qx_1} e^{qx_2} dx_1 dx_2,$$

where p and q are any positive numbers. We shall study some growth and asymptotic properties of $I_p(\sigma_1, \sigma_2)$ and $m_{p,q}(\sigma_1, \sigma_2)$.

2. Theorem 1. *If $f(s_1, s_2)$, $f \in F$ is of finite order, then*

$$(2.1) \quad \log M(\sigma_1, \sigma_2) \sim \log \mu(\sigma_1, \sigma_2).$$

PROOF. From the definition of $M(\sigma_1, \sigma_2)$, we have

$$\begin{aligned} M(\sigma_1, \sigma_2) &\leq \sum_{m,n=1}^{\infty} |a_{m,n}| \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) \\ &= \left[\sum_{m=1}^{m_0} \sum_{n=1}^{n_0} + \sum_{m=m_0+1}^{\infty} \sum_{n=1}^{n_0} + \sum_{m=1}^{m_0} \sum_{n=n_0+1}^{\infty} + \sum_{m=m_0+1}^{\infty} \sum_{n=n_0+1}^{\infty} \right] \times \\ &\quad \times |a_{m,n}| \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) \\ (2.2) \quad &= \sum_1 + \sum_2 + \sum_3 + \sum_4, \quad (\text{say}) \end{aligned}$$

Now

$$\begin{aligned}
\sum_1 &\leq A \exp(\sigma_1 \lambda_{m_0} + \sigma_2 \mu_{n_0}) \\
\sum_2 &= \sum_{m=m_0+1}^{\infty} \sum_{n=1}^{n_0} |a_{m,n}| \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) \\
&< B \exp(\sigma_2 \mu_{n_0}) \sum_{m=m_0+1}^{\infty} |a_{m,n}| \exp(\sigma_1 \lambda_m) \\
&= B \exp(\sigma_2 \mu_{n_0}) \sum_{m=m_0+1}^{\infty} |a_{m,n}| \exp(\sigma_1 + 2\varepsilon) \lambda_m \exp(-2\varepsilon \lambda_m).
\end{aligned}$$

Similarly

$$\sum_3 < C \exp(\sigma_1 \lambda_{m_0}) \sum_{n=n_0+1}^{\infty} |a_{m,n}| \exp(\sigma_2 + 2\varepsilon) \mu_n \exp(-2\varepsilon \mu_n).$$

Also

$$\begin{aligned}
\sum_4 &= \sum_{m=m_0+1}^{\infty} \sum_{n=n_0+1}^{\infty} |a_{m,n}| \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) \\
&= \sum_{m>m_0} \sum_{n>n_0} |a_{m,n}| \exp\{(\sigma_1 + 2\varepsilon) \lambda_m\} \exp\{(\sigma_2 + 2\varepsilon) \mu_n\} \\
&\quad \times \exp(-2\varepsilon \lambda_m) \exp(-2\varepsilon \mu_n).
\end{aligned}$$

Using (1.2), we have for a given $\varepsilon > 0$,

$$\begin{aligned}
\sum_2 &< B \exp(\sigma_2 \mu_{n_0}) \sum_{m=m_0+1}^{\infty} |a_{m,n}| \exp\{(\sigma_1 + 2\varepsilon) \lambda_m\} \frac{1}{m^2} \\
\sum_3 &< C \exp(\sigma_1 \lambda_{m_0}) \sum_{m=n_0+1}^{\infty} |a_{m,n}| \exp\{(\sigma_2 + 2\varepsilon) \mu_n\} \frac{1}{n^2}
\end{aligned}$$

and

$$\begin{aligned}
\sum_4 &< \sum_{m>m_0} \sum_{n>n_0} |a_{m,n}| \exp\{(\sigma_1 + 2\varepsilon) \lambda_m\} \exp\{(\sigma_2 + 2\varepsilon) \mu_n\} \frac{1}{m^2 n^2} \\
&< K_1 \mu(\sigma_1 + 2\varepsilon, \sigma_2 + 2\varepsilon) \sum_{m>m_0} \sum_{n>n_0} \frac{1}{m^2 n^2}.
\end{aligned}$$

Hence (2.2) gives

$$\begin{aligned}
(2.3) \quad M(\sigma_1, \sigma_2) &\leq A \exp(\sigma_1 \lambda_{m_0} + \sigma_2 \mu_{n_0}) \\
&\quad + B \exp(\sigma_2 \mu_{n_0}) \sum_{m=m_0+1}^{\infty} |a_{m,n}| \exp\{(\sigma_1 + 2\varepsilon) \lambda_m\} \frac{1}{m^2} \\
&\quad + C \exp(\sigma_1 \lambda_{m_0}) \sum_{n=n_0+1}^{\infty} |a_{m,n}| \exp\{(\sigma_2 + 2\varepsilon) \mu_n\} \frac{1}{n^2} \\
&\quad + K_1 \mu(\sigma_1 + 2\varepsilon, \sigma_2 + 2\varepsilon) \sum_{m>m_0} \sum_{n>n_0} \frac{1}{m^2 n^2} \\
(2.4) \quad &\leq 0(\mu(\sigma_1 + 2\varepsilon, \sigma_2 + 2\varepsilon)).
\end{aligned}$$

We also note that $\mu(\sigma_1, \sigma_2) \leq M(\sigma_1, \sigma_2)$. Hence $\log M(\sigma_1, \sigma_2) \sim \log \mu(\sigma_1, \sigma_2)$.

3. Theorem 2. *If $f(s_1, s_2)$, $f \in F$ is of finite order, then for $p \geq 1$,*

$$(3.1) \quad \log I_p(\sigma_1, \sigma_2) \sim p \log M(\sigma_1, \sigma_2).$$

We first prove the following

Lemma 1. *For $f(s_1, s_2)$, $f \in F$*

$$\begin{aligned}
(3.2) \quad a_{m,n} \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) &= \lim_{(T_1, T_2) \rightarrow \infty} \frac{1}{4T_1 T_2} \times \\
&\times \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} [\exp(-i(t_1 \lambda_m + t_2 \mu_n)) \times f(\sigma_1 + it_1, \sigma_2 + it_2)] dt_1 dt_2.
\end{aligned}$$

PROOF. We have

$$\begin{aligned}
f(s_1, s_2) &= f(\sigma_1 + it_1, \sigma_2 + it_2) \\
&= \sum_{M, N=1}^{\infty} a_{M,N} \exp\{(\sigma_1 + it_1) \lambda_M + (\sigma_2 + it_2) \mu_N\} \\
&= a_{m,n} \exp\{(\sigma_1 + it_1) \lambda_m + (\sigma_2 + it_2) \mu_n\} \\
&\quad + \sum_{M \neq m} \sum_{N \neq n} a_{M,N} \exp\{(\sigma_1 + it_1) \lambda_M + (\sigma_2 + it_2) \mu_N\}
\end{aligned}$$

Also

$$\exp\{-i(t_1 \lambda_m + t_2 \mu_n)\} f(\sigma_1 + it_1, \sigma_2 + it_2) =$$

$$\begin{aligned}
&= a_{m,n} \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) + \sum_{M \neq m} \sum_{N \neq n} a_{M,N} \exp\{(\sigma_1 \lambda_M + \sigma_2 \mu_N)\} \times \\
&\quad \times \exp\{it_1(\lambda_M - \lambda_m) + it_2(\mu_N - \mu_n)\}.
\end{aligned}$$

Since the series on the right is uniformly convergent for any finite t_1 and t_2 ranges, therefore we may integrate term by term for finite t_1 and t_2 . Hence on integration, all the terms, for which $m \neq M$, $n \neq N$, after dividing by $T_1 T_2$ vanish as $T_1, T_2 \rightarrow \infty$ and we obtain

$$\begin{aligned}
\lim_{(T_1, T_2) \rightarrow \infty} \frac{1}{4T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \exp\{-i(t_1 \lambda_m + t_2 \mu_n)\} f(\sigma_1 + it_1, \sigma_2 + it_2) dt_1 dt_2 \\
= a_{m,n} \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n).
\end{aligned}$$

PROOF of Theorem 2. From (1.4), we have

$$\begin{aligned}
I_p(\sigma_1, \sigma_2) &= \lim_{(T_1, T_2) \rightarrow \infty} \frac{1}{4T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2 \\
(3.3) \quad &\leq \{M(\sigma_1, \sigma_2)\}^p.
\end{aligned}$$

Also from Lemma 1, we get

$$(3.4) \quad \mu(\sigma_1, \sigma_2) \leq \lim_{(T_1, T_2) \rightarrow \infty} \frac{1}{4T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)| dt_1 dt_2.$$

Applying Hölder's inequality for $p > 1$, we have

$$\begin{aligned}
&\int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)| dt_1 dt_2 \\
&\leq \int_{-T_1}^{T_1} \left[\left\{ \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_2 \right\}^{1/p} \left\{ \int_{-T_2}^{T_2} dt_2 \right\}^{1-\frac{1}{p}} \right] dt_1 \\
&\leq \left[\int_{-T_1}^{T_1} \left\{ \left(\int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_2 \right)^{\frac{1}{p}} \right\}^p dt_1 \right]^{\frac{1}{p}} \times
\end{aligned}$$

$$\begin{aligned}
& \times \left[\int_{-T_1}^{T_1} \left\{ \left(\int_{-T_2}^{T_2} dt_2 \right)^{1-\frac{1}{p}} \right\}^{\frac{p}{p-1}} dt_1 \right]^{1-\frac{1}{p}} \\
& = \left[\int_{-T_1}^{T_1} \left\{ \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_2 \right\} dt_1 \right]^{\frac{1}{p}} \left[\left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} dt_2 \right\} dt_1 \right]^{1-\frac{1}{p}} \\
& = \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2 \right\}^{\frac{1}{p}} \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} dt_1 dt_2 \right\}^{1-\frac{1}{p}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3.5) \quad \mu(\sigma_1, \sigma_2) & \leq \lim_{(T_1, T_2) \rightarrow \infty} \frac{1}{4T_1 T_2} \\
& \times \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2 \right\}^{\frac{1}{p}} \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} dt_1 dt_2 \right\}^{1-\frac{1}{p}} \\
(3.6) \quad & = \lim_{(T_1, T_2) \rightarrow \infty} \left\{ \frac{1}{4T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2 \right\}^{\frac{1}{p}} \\
& = \left\{ I_p(\sigma_1, \sigma_2) \right\}^{\frac{1}{p}}.
\end{aligned}$$

For, $p = 1$, we have from (4.4)

$$(3.7) \quad \mu(\sigma_1, \sigma_2) \leq I_1(\sigma_1, \sigma_2)$$

Hence for $p \geq 1$, we have

$$(3.8) \quad \mu(\sigma_1, \sigma_2) \leq \{I_p(\sigma_1, \sigma_2)\}^{\frac{1}{p}} \leq M(\sigma_1, \sigma_2),$$

and using (2.1), we get the theorem.

Corollary 1. *If $f(s_1, s_2)$, $f \in F$, is of finite order $\rho = (\rho_1, \rho_2) \gg (0, 0)$, then for $p \geq 1$*

$$\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log \log I_p(\sigma_1, \sigma_2)}{\log(e^{\sigma_1 \rho_1} + e^{\sigma_2 \rho_2})} \right\} = 1.$$

Corollary 2. *If $f(s_1, s_2)$, $f \in F$ is of finite order $\rho = (\rho_1, \rho_2) \gg (0, 0)$ and type $\tau = (\tau_1, \tau_2) \gg (0, 0)$, then for $p \geq 1$*

$$\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log I_p(\sigma_1, \sigma_2)}{(\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2})} \right\} = p .$$

4. Theorem 3. *If $f(s_1, s_2)$, $f \in F$ is of finite order $\rho = (\rho_1, \rho_2) \gg (0, 0)$ then for $p \geq 1$*

$$(4.1) \quad \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log \log m_{p,q}(\sigma_1, \sigma_2)}{\log(e^{\sigma_1 \rho_1} + e^{\sigma_2 \rho_2})} \right\} = 1 .$$

PROOF. We use (3.8) in (1.5) and then with the help of (2.4) we get

$$(4.2) \quad \begin{aligned} m_{p,q}(\sigma_1, \sigma_2) &= \frac{4}{e^{q\sigma_1} e^{q\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} I_p(x_1, x_2) e^{qx_1} e^{qx_2} dx_1 dx_2 \\ &\leq \frac{4}{e^{q\sigma_1} e^{q\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} \{M(x_1, x_2)\}^p e^{qx_1} e^{qx_2} dx_1 dx_2 \\ &\leq \frac{4}{e^{q\sigma_1} e^{q\sigma_2}} \frac{\{M(\sigma_1, \sigma_2)\}^p}{q^2} (e^{q\sigma_1} - 1)(e^{q\sigma_2} - 1) \\ &= \frac{4}{q^2} \{M(\sigma_1, \sigma_2)\}^p (1 - e^{-q\sigma_1})(1 - e^{-q\sigma_2}) \\ (4.3) \quad &< \frac{4}{q^2} \{\mu(\sigma_1 + 2\varepsilon, \sigma_2 + 2\varepsilon)\}^p (1 - e^{-q\sigma_1})(1 - e^{-q\sigma_2}) \end{aligned}$$

Also, using (3.8) in (1.5), we have for $h > 0$

$$(4.4) \quad \begin{aligned} m_{p,q}(\sigma_1 + \frac{h}{\rho_1}, \sigma_2 + \frac{h}{\rho_2}) &= \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_0^{\sigma_1 + \frac{h}{\rho_1}} \int_0^{\sigma_2 + \frac{h}{\rho_2}} I_p(x_1, x_2) e^{qx_1} e^{qx_2} dx_1 dx_2 \\ &\geq \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_0^{\sigma_1 + \frac{h}{\rho_1}} \int_0^{\sigma_2 + \frac{h}{\rho_2}} \{\mu(x_1, x_2)\}^p e^{qx_1} e^{qx_2} dx_1 dx_2 \\ &\geq \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_{\sigma_1}^{\sigma_1 + \frac{h}{\rho_1}} \int_{\sigma_2}^{\sigma_2 + \frac{h}{\rho_2}} \{\mu(x_1, x_2)\}^p e^{qx_1} e^{qx_2} dx_1 dx_2 \end{aligned}$$

$$\begin{aligned}
&\geq \frac{4\{\mu(\sigma_1, \sigma_2)\}^p}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \times \frac{1}{q^2} \left\{ (e^{q(\sigma_1 + \frac{h}{\rho_1})} - e^{q\sigma_1}) \right\} \times \\
&\quad \times \left\{ e^{q(\sigma_2 + \frac{h}{\rho_2})} - e^{q\sigma_2} \right\} \\
&= \frac{4}{q^2} \{\mu(\sigma_1, \sigma_2)\}^p (1 - e^{-q\frac{h}{\rho_1}})(1 - e^{-q\frac{h}{\rho_2}}) .
\end{aligned}$$

From (5.3) and (5.4), we obtain

$$\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log \log m_{p,q}(\sigma_1, \sigma_2)}{\log(e^{\sigma_1 \rho_1} + e^{\sigma_2 \rho_2})} \right\} = \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log \log \mu(\sigma_1, \sigma_2)}{\log(e^{\sigma_1 \rho_1} + e^{\sigma_2 \rho_2})} \right\}$$

Using (2.1) and Lemma A, we get

$$\begin{aligned}
&\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log \log m_{p,q}(\sigma_1, \sigma_2)}{\log(e^{\sigma_1 \rho_1} + e^{\sigma_2 \rho_2})} \right\} \\
&= \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log \log M(\sigma_1, \sigma_2)}{\log(e^{\sigma_1 \rho_1} + e^{\sigma_2 \rho_2})} \right\} = 1
\end{aligned}$$

and hence the result.

5. Theorem 4. *If $f(s_1, s_2)$, $f \in F$ is of finite order $\rho = (\rho_1, \rho_2) \gg (0, 0)$ and type $\tau = (\tau_1, \tau_2) \gg (0, 0)$ then for $p \geq 1$*

$$(5.1) \quad \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left\{ \frac{\log m_{p,q}(\sigma_1, \sigma_2)}{(\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2})} \right\} = p .$$

PROOF. We have

$$\begin{aligned}
(5.2) \quad m_{p,q}(\sigma_1, \sigma_2) &= \frac{4}{e^{q\sigma_1} e^{q\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} I_p(x_1, x_2) e^{qx_1} e^{qx_2} dx_1 dx_2 \\
&\leq \frac{4}{e^{q\sigma_1} e^{q\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} \{M(x_1, x_2)\}^p e^{qx_1} e^{qx_2} dx_1 dx_2 \\
&\leq \frac{4}{q^2} \{M(\sigma_1, \sigma_2)\}^p (1 - e^{-q\sigma_1})(1 - e^{-q\sigma_2}) .
\end{aligned}$$

Therefore

$$\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log m_{p,q}(\sigma_1, \sigma_2)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right] \leq p \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log M(\sigma_1, \sigma_2)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right]$$

Also

$$m_{p,q}\left(\sigma_1 + \frac{h}{\rho_1}, \sigma_2 + \frac{h}{\rho_2}\right) =$$

$$\begin{aligned}
&= \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_0^{\sigma_1 + \frac{h}{\rho_1}} \int_0^{\sigma_2 + \frac{h}{\rho_2}} I_p(x_1, x_2) e^{qx_1} e^{qx_2} dx_1 dx_2 \\
&\geq \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_0^{\sigma_1 + \frac{h}{\rho_1}} \int_0^{\sigma_2 + \frac{h}{\rho_2}} \{\mu(x_1, x_2)\}^p e^{qx_1} e^{qx_2} dx_1 dx_2 \\
&\geq \frac{4}{q^2} \{\mu(\sigma_1, \sigma_2)\}^p (1 - e^{-q\frac{h}{\rho_1}})(1 - e^{-q\frac{h}{\rho_2}}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&e^h \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log m_{p,q}(\sigma_1 + \frac{h}{\rho_1}, \sigma_2 + \frac{h}{\rho_2})}{\tau_1 e^{\rho_1(\sigma_1 + \frac{h}{\rho_1})} + \tau_2 e^{\rho_2(\sigma_2 + \frac{h}{\rho_2})}} \right] \\
&\geq p \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log \mu(\sigma_1, \sigma_2)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right] \\
\text{or } &e^h \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log m_{p,q}(\sigma_1, \sigma_2)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right] \\
&\geq p \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log \mu(\sigma_1, \sigma_2)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right].
\end{aligned}$$

Since h is arbitrary, we have

$$\begin{aligned}
(5.3) \quad &\lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log m_{p,q}(\sigma_1, \sigma_2)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right] \\
&\geq p \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \text{Sup} \left[\frac{\log \mu(\sigma_1, \sigma_2)}{\tau_1 e^{\sigma_1 \rho_1} + \tau_2 e^{\sigma_2 \rho_2}} \right].
\end{aligned}$$

Also $\log M(\sigma_1, \sigma_2) \sim \log \mu(\sigma_1, \sigma_2)$ and therefore from (5.2) and (5.3), we get the theorem.

6. Theorem 5. For $f(s_1, s_2)$, $f \in F$ and $p \geq 1$,

$$(6.1) \quad \lim_{(\sigma_1, \sigma_2) \rightarrow \infty} \left[\frac{1}{e^{q\sigma_1(1-\alpha_1)} e^{q\sigma_2(1-\alpha_2)} m_{p,q}(\sigma_1, \sigma_2) - m_{p,q}(\alpha_1 \sigma_1, \alpha_2 \sigma_2)} \right] = 0$$

where α_1, α_2 ($0 < \alpha_1, \alpha_2 < 1$) are constants.

We first prove the following lemma:

Lemma 2. Let $f(s_1, s_2)$, $f \in F$, then for $(0 < \sigma'_1 < \bar{\sigma}_1 < \sigma_1)$, and $(0 < \sigma'_2 < \bar{\sigma}_2 < \sigma_2)$,

$$\begin{aligned}
& \left[\{\mu(\bar{\sigma}_1, \sigma'_2)\}^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\bar{\sigma}_2} - e^{q\sigma'_2}) \right. \\
& \quad + \{\mu(\sigma'_1, \bar{\sigma}_2)\}^p (e^{q\bar{\sigma}_1} - e^{q\sigma'_1})(e^{q\sigma_2} - e^{q\bar{\sigma}_2}) \\
& \quad \left. + \{\mu(\bar{\sigma}_1, \bar{\sigma}_2)\}^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\sigma_2} - e^{q\bar{\sigma}_2}) \right] \\
& \leq \left(\frac{q}{2} \right)^2 \left[e^{q\sigma_1+q\sigma_2} m_{p,q}(\sigma_1, \sigma_2) - e^{q\bar{\sigma}_1+q\bar{\sigma}_2} m_{p,q}(\bar{\sigma}_1, \bar{\sigma}_2) \right] \\
& \leq \left[\{M(\sigma_1, \bar{\sigma}_2)\}^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\bar{\sigma}_2} - 1) \right. \\
& \quad + \{M(\bar{\sigma}_1, \sigma_2)\}^p (e^{q\bar{\sigma}_1} - 1)(e^{q\sigma_2} - e^{q\bar{\sigma}_2}) \\
& \quad \left. + \{M(\sigma_1, \sigma_2)\}^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\sigma_2} - e^{q\bar{\sigma}_2}) \right],
\end{aligned}$$

where $p \geq 1$ and q is any positive number.

PROOF of Lemma 2. From (1.3), we have

$$\begin{aligned}
& e^{q\sigma_1+q\sigma_2} m_{p,q}(\sigma_1, \sigma_2) - e^{q\bar{\sigma}_1+q\bar{\sigma}_2} m_{p,q}(\bar{\sigma}_1, \bar{\sigma}_2) \\
& = 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_0^{\bar{\sigma}_2} I_p(x_1, x_2) e^{qx_1+qx_2} dx_1 dx_2 + 4 \int_0^{\bar{\sigma}_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_p(x_1, x_2) e^{qx_1+qx_2} dx_1 dx_2 \\
& \quad + 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_p(x_1, x_2) e^{qx_1+qx_2} dx_1 dx_2
\end{aligned}$$

Hence, using (3.8), we have for $p \geq 1$

$$\begin{aligned}
(6.2) \quad & e^{q\sigma_1+q\sigma_2} m_{p,q}(\sigma_1, \sigma_2) - e^{q\bar{\sigma}_1+q\bar{\sigma}_2} m_{p,q}(\bar{\sigma}_1, \bar{\sigma}_2) \\
& \leq 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_0^{\bar{\sigma}_2} \{M(x_1, x_2)\}^p e^{qx_1+qx_2} dx_1 dx_2 + \\
& \quad + 4 \int_0^{\bar{\sigma}_1} \int_{\bar{\sigma}_2}^{\sigma_2} \{M(x_1, x_2)\}^p e^{qx_1+qx_2} dx_1 dx_2 \\
& \quad + 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\bar{\sigma}_2}^{\sigma_2} \{M(x_1, x_2)\}^p e^{qx_1+qx_2} dx_1 dx_2 \\
& \leq \frac{4}{q^2} [M(\sigma_1, \bar{\sigma}_2)]^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\bar{\sigma}_2} - 1)
\end{aligned}$$

$$\begin{aligned}
& + \frac{4}{q^2} [M(\bar{\sigma}_1, \sigma_2)]^p (e^{q\bar{\sigma}_1} - 1)(e^{q\sigma_2} - e^{q\bar{\sigma}_2}) \\
& + \frac{4}{q^2} [M(\sigma_1, \sigma_2)]^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\sigma_2} - e^{q\bar{\sigma}_2}).
\end{aligned}$$

Also

$$\begin{aligned}
(6.3) \quad & e^{q\sigma_1+q\sigma_2} m_{p,q}(\sigma_1, \sigma_2) - e^{q\bar{\sigma}_1+q\bar{\sigma}_2} m_{p,q}(\bar{\sigma}_1, \bar{\sigma}_2) \\
& \geq 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\sigma'_2}^{\bar{\sigma}_2} [\mu(x_1, x_2)]^p e^{qx_1+qx_2} dx_1 dx_2 \\
& \quad + 4 \int_{\sigma'_1}^{\bar{\sigma}_1} \int_{\bar{\sigma}_2}^{\sigma_2} [\mu(x_1, x_2)]^p e^{qx_1+qx_2} dx_1 dx_2 \\
& \quad + 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\bar{\sigma}_2}^{\sigma_2} [\mu(x_1, x_2)]^p e^{qx_1+qx_2} dx_1 dx_2 \\
& \geq \frac{4}{q^2} [\mu(\bar{\sigma}_1, \sigma'_2)]^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\bar{\sigma}_2} - e^{q\sigma'_2}) \\
& \quad + \frac{4}{q^2} [\mu(\sigma'_1, \bar{\sigma}_2)]^p (e^{q\bar{\sigma}_1} - e^{q\sigma'_1})(e^{q\sigma_2} - e^{q\bar{\sigma}_2}) \\
& \quad + \frac{4}{q^2} [\mu(\bar{\sigma}_1, \bar{\sigma}_2)]^p (e^{q\sigma_1} - e^{q\bar{\sigma}_1})(e^{q\sigma_2} - e^{q\bar{\sigma}_2}).
\end{aligned}$$

Combining (7.2) and (7.3) we obtain the lemma.

PROOF of Theorem 5. If we put

$$\bar{\sigma}_1 = \alpha_1 \sigma_1, \quad \sigma'_1 = \beta_1 \sigma_1, \quad \bar{\sigma}_2 = \alpha_2 \sigma_2, \quad \sigma'_2 = \beta_2 \sigma_2,$$

where $\beta_1 < \alpha_1$, $\beta_2 < \alpha_2$ in Lemma 2, we get

$$\begin{aligned}
& \left[4 \{ \mu(\alpha_1 \sigma_1, \beta_2 \sigma_2) \}^p (e^{q\sigma_1} - e^{q\alpha_1 \sigma_1})(e^{q\alpha_2 \sigma_2} - e^{q\beta_2 \sigma_2}) \right. \\
& \quad + 4 \{ \mu(\beta_1 \sigma_1, \alpha_2 \sigma_2) \}^p (e^{q\alpha_1 \sigma_1} - e^{q\beta_1 \sigma_1})(e^{q\sigma_2} - e^{q\alpha_2 \sigma_2}) \\
& \quad \left. + 4 \{ \mu(\alpha_1 \sigma_1, \alpha_2 \sigma_2) \}^p (e^{q\sigma_1} - e^{q\alpha_1 \sigma_1})(e^{q\sigma_2} - e^{q\alpha_2 \sigma_2}) \right] \\
& \leq q^2 \left[e^{q\sigma_1+q\sigma_2} m_{p,q}(\sigma_1, \sigma_2) - e^{q\alpha_1 \sigma_1+q\alpha_2 \sigma_2} m_{p,q}(\alpha_1 \sigma_1, \alpha_2 \sigma_2) \right] \\
& \leq \left[4 \{ M(\sigma_1, \alpha_2 \sigma_2) \}^p (e^{q\sigma_1} - e^{q\alpha_1 \sigma_1})(e^{q\alpha_2 \sigma_2} - 1) \right]
\end{aligned}$$

$$\begin{aligned}
& + 4\{M(\alpha_1\sigma_1, \sigma_2)\}^p(e^{q\alpha_1\sigma_1} - 1)(e^{q\sigma_2} - e^{q\alpha_2\sigma_2}) \\
& + 4\{M(\sigma_1, \sigma_2)\}^p(e^{q\sigma_1} - e^{q\alpha_1\sigma_1})(e^{q\sigma_2} - e^{q\alpha_2\sigma_2}) \Big]
\end{aligned}$$

Dividing by $e^{q\alpha_1\sigma_1} e^{q\alpha_2\sigma_2}$, we obtain

$$\begin{aligned}
& \left[4\{\mu(\alpha_1\sigma_1, \beta_2\sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)} - 1)(1 - e^{q\sigma_2(\beta_2-\alpha_2)}) \right. \\
& \quad + 4\{\mu(\beta_1\sigma_1, \alpha_2\sigma_2)\}^p(e^{q\sigma_2(1-\alpha_2)} - 1)(1 - e^{q\sigma_1(\beta_1-\alpha_1)}) \\
& \quad \left. + 4\{\mu(\alpha_1\sigma_1, \alpha_2\sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)} - 1)(e^{q\sigma_2(1-\alpha_2)} - 1) \right] \\
& \leq q^2 \left[e^{q\sigma_1(1-\alpha_1)} e^{q\sigma_2(1-\alpha_2)} m_{p,q}(\sigma_1, \sigma_2) - m_{p,q}(\alpha_1\sigma_1, \alpha_2\sigma_2) \right] \\
& \leq \left[4\{M(\sigma_1, \alpha_2\sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)} - 1)(1 - e^{-q\alpha_2\sigma_2}) \right. \\
& \quad + 4\{M(\alpha_1\sigma_1, \sigma_2)\}^p(1 - e^{-q\alpha_1\sigma_1})(e^{q\sigma_2(1-\alpha_2)} - 1) \\
& \quad \left. + 4\{M(\sigma_1, \sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)} - 1)(e^{q\sigma_2(1-\alpha_2)} - 1) \right]
\end{aligned}$$

Taking limits on both the sides, Theorem 5 follows from the above inequalities.

I take this opportunity to express my thanks to Dr. S. N. SRIVASTAVA for his valuable suggestions and guidance in the preparation of this paper.

References

- [1] A. I. JANUSAUSKAS, Elementary theorem on the convergence of Double Dirichlet Series, *Dokl. Akad. Nauk SSSR* **234**, 1.
- [2] P. K. SARKAR, On Order and type of an Entire function in several complex variables represented by Dirichlet series, *Bull. Cal. Math. Soc.* **74** (1982), 99-112.

VINITA VIJAI
DEPARTMENT OF MATHEMATICS AND ASTRONOMY
UNIVERSITY OF LUCKNOW
LUCKNOW
INDIA

(Received August 28, 1989)