

Analytic solutions of Schröder equation

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Let E be a Banach space over the field of complex numbers $\mathbb{C}$.

Definition 1. ([1]) A mapping $f: E \rightarrow E$ is called a homogeneous polynomial of degree n if and only if there exists a n -linear symmetrical mapping $\bar{f}: E^n \rightarrow E$ such that

$$f(x) = \bar{f}(x, \dots, x), \quad x \in E.$$

We say that $\bar{f}$ is associated with f .

Let $P_n(E)$ be a space all continuous and homogeneous polynomials of E into E of degree n ($P_0(E)$ denotes the set of constant mappings from E into E).

We shall investigate local analytic solutions of the Schröder equation

$$(1) \quad \varphi[f(x)] = A\varphi(x),$$

where a function f of the type $f: E \rightarrow E$ and $A \in P_1(E)$ are given and φ of the type $\varphi: E \rightarrow E$ is unknown function.

We assume that

(I) The function f is analytic in open set $G \subset E$, $0 \in G$, and $f(0) = 0$.

(II) The mapping $A := f'(0)$ is invertible of the space E onto E and

$$(2) \quad \|A\|^2 \|A^{-1}\| < 1.$$

Definition 2. ([1]) A function $f: G \rightarrow E$ is called analytic in G if for every point $x \in G$ there exist $f_n \in P_n(E)$, $n = 0, 1, 2, \dots$, such that

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h)$$

for all h in a neighbourhood of $0 \in E$.

For every n there exists n -th Frechét derivative $D^n f(x)$ of f at $x \in G$ and

$$D^n f(x) = n! \bar{f}_n, \quad \text{for } n = 0, 1, 2, \dots,$$

where $\bar{f}_n$ is associated with f (cf. [1]).

It follows by $A = A^{-1}AA$ and (2) that

$$(3) \quad s := \|A\| \leq \|A^{-1}\| \|A\|^2 < 1.$$

According to (2) there exists a number σ , $s < \sigma < 1$, such that

$$(4) \quad \|A^{-1}\| < \frac{1}{\sigma^2}.$$

Lemma 1. *There exist numbers $R > 0$ and $K > 0$ such that*

$$\|f(x) - Ax\| \leq K\|x\|^2 \quad \text{for } \|x\| \leq R.$$

PROOF. By virtue of hypothesis (I) and by [2] (Th. 6.2) there exist numbers $M > 0$ and R_0 , $0 < R_0 < 1$ such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for } \|x\| \leq R_0,$$

$$\{x: \|x\| \leq R_0\} \subset G$$

and

$$\|f(x)\| \leq M \quad \text{for } \|x\| \leq R_0.$$

For $\|x\| < \frac{1}{2} R_0$ we have

$$\|f_n(x)\| \leq \frac{M}{2^n}, \quad n \geq 1.$$

Hence

$$\sum_{n=2}^{\infty} \|f_n(x)\| \leq \frac{1}{2} M \quad \text{for } \|x\| \leq \frac{1}{2} R_0.$$

Let R be a fixed number such that

$$0 < R < \frac{1}{2} R_0.$$

We get

$$\sum_{n=2}^{\infty} \|f_n(x)\| = \sum_{n=2}^{\infty} \left(\frac{\|x\|}{R} \right)^n \left\| f_n \left(\frac{x}{\|x\|} R \right) \right\| \leq \frac{\|x\|^2}{2R^2} M$$

for $\|x\| \leq R$. Thus

$$\|f(x) - Ax\| \leq \frac{M}{2R^2} \|x\|^2 \quad \text{for } \|x\| \leq R.$$

Putting $K := \frac{M}{2R^2}$ we get the assertion of this lemma.

Lemma 2. *There exist positive numbers r and ϑ , $s < \vartheta < \sigma$, such that*

$$(5) \quad \|f(x)\| < \vartheta \|x\| \quad \text{for } \|x\| \leq r.$$

PROOF. We have by Lemma 1

$$\|f(x)\| \leq \|f(x) - Ax\| + \|Ax\| \leq K\|x\|^2 + s\|x\| = (K\|x\| + s)\|x\|.$$

Hence, it follows that there exists sufficiently small number $r > 0$ such that inequality (5) is fulfilled for $\|x\| \leq r$.

In virtue of (5) the function f transforms the ball $\{x: \|x\| < r\}$ into itself, so we may define n -th iterate of the function f :

$$f^0(x) = x, \quad f^{n+1}(x) = f(f^n(x)), \quad n = 0, 1, 2, \dots$$

The functions f^n , are analytic (cf. [2]) for $\|x\| < r$.

Theorem 1. *If hypotheses (I) and (II) are fulfilled, then the sequence $\{A^{-n}f^n(x)\}$ is uniformly convergent in some neighbourhood of zero and the function*

$$(6) \quad \varphi(x) := \lim_{n \rightarrow \infty} A^{-n}f^n(x)$$

is the analytic solution of Schröder equation (1).

PROOF. By lemmas and by (4) we have

$$\begin{aligned} \|A^{-(n+1)}f^{n+1}(x) - A^{-n}f^n(x)\| &= \|A^{-(n+1)}(f^{n+1}(x) - Af^n(x))\| \leq \\ &\leq \|A^{-1}\|^{n+1} \|f^{n+1}(x) - Af^n(x)\| \leq \frac{K}{\sigma^{2(n+1)}} \|f^n(x)\|^2 \leq \frac{K}{\sigma^2} r^2 \left(\frac{\vartheta}{\sigma}\right)^{2n} \end{aligned}$$

for $\|x\| \leq r$. Hence, the sequence $\{A^{-n}f^n(x)\}$ is uniformly convergent in the set $\{x: \|x\| \leq r\}$. The functions $A^{-n}f^n(x)$ are analytic. Thus, the function φ given by (6) is analytic in $\{x: \|x\| < r\}$ (cf. [1]). Since

$$\varphi[f(x)] = \lim_{n \rightarrow \infty} A^{-n}f^{n+1}(x) = \lim_{n \rightarrow \infty} A^{-(n+1)}Af^{n+1}(x) = A\varphi(x),$$

thus φ is the solution of (1). The proof is completed.

It is known that, if $\|A\| < 1$, that the mapping $A - I((A - I)(x) = Ax - x)$ is invertible. Therefore if φ is a solution of equation (1), then $\varphi(0) = 0$.

Theorem 2. *If hypotheses (I) and (II) are fulfilled, then the solution of Schröder equation given by (6) is invertible in an neighbourhood of zero.*

PROOF. At first, we shall prove that the sequence

$$\varphi'_n(x) = (A^{-n}f^n)'(x)$$

is uniformly convergent in some neighbourhood of zero. We have

$$\begin{aligned} \|\varphi'_{n+1}(x) - \varphi'_n(x)\| &= \|A^{-(n+1)}f'(f^n(x)) \circ \dots \circ f'(x) - A^{-n}f'(f^{n-1}(x)) \circ \dots \circ f'(x)\| \\ &\leq \|A^{-(n+1)}\| \|(f'(f^n(x)) - A)(f'(f^{n-1}(x)) \circ \dots \circ f'(x))\| \leq \\ &\leq \|A^{-1}\|^{n+1} \|f'(f^n(x)) - f'(0)\| \|f'(f^{n-1}(x))\| \dots \|f'(x)\| \end{aligned}$$

for $\|x\| \leq r$, where r is defined in Lemma 2. There exists a number $r_1 \leq r$ such that

$$\|f'(x)\| < \vartheta \quad \text{for } \|x\| \leq r_1,$$

where ϑ is the constant from Lemma 2. The function $f''(x)$ is analytic therefore (cf. [2]) there exist positive numbers M and $r_2 \leq r_1$ such that

$$\|f''(x)\| \leq M \quad \text{for } \|x\| \leq r_2.$$

By mean value theorem we get

$$\|f'(f^n(x)) - f'(0)\| \leq M\vartheta^n \|x\| \quad \text{for } \|x\| \leq r_2.$$

Thus, we have

$$\|\varphi'_{n+1}(x) - \varphi'_n(x)\| \leq \frac{Mr_2}{\sigma^2} \left(\frac{\vartheta}{\sigma}\right)^{2n}.$$

Hence the uniform convergence of $\{\varphi'_n(x)\}$ for $\|x\| \leq r_2$ follows. In virtue of Th. 8.6.3 from [3] we have

$$\varphi'(x) = \lim_{n \rightarrow \infty} A^{-n} f'(f^{n-1}(x)) \circ \dots \circ f'(x), \quad \text{and} \quad \varphi'(0) = I.$$

Hence, by Th. 10.2.5. from [3], the function φ is invertible in a neighbourhood of zero.

Theorem 3. *Let hypotheses (I) and (II) be fulfilled. If function ψ is an analytic solution of Schröder equation (1) in a neighbourhood U of zero, then there exists linear and continuous mapping $S: E \rightarrow E$ such that*

$$SA = AS \quad \text{and} \quad \psi(x) = S\varphi(x),$$

where φ is given by (6).

PROOF. We have $\varphi \circ f = A\varphi$, $\psi \circ f = A\psi$, $f = \varphi^{-1} \circ A\varphi$, $f \circ \varphi^{-1} = \varphi^{-1} \circ A$ in U . Let $L := \psi \circ \varphi^{-1}$. Hence

$$AL = A\psi \circ \varphi^{-1} = \psi \circ f \circ \varphi^{-1} = \psi \circ \varphi^{-1} \circ A = L \circ A.$$

Thus, the mapping L is analytic (cf. [4]) and commutable with A . There exists a neighbourhood V of zero such that

$$L(x) = S(x) + \sum_{k=2}^{\infty} S_k(x), \quad x \in V \subset U,$$

where $S_k \in P_k(E)$. We have equality

$$S(A(x)) + \sum_{k=2}^{\infty} S_k(A(x)) = AS(x) + \sum_{k=2}^{\infty} AS_k(x)$$

because $LA = AL$. Hence

$$SA = AS \quad \text{and} \quad S_k A = AS_k, \quad k \geq 2.$$

From the last equality we get

$$\|S_k\| \leq \|A^{-1}\| \|S_k\| \|A\|^k, \quad k \geq 2.$$

By virtue of hypothesis (II) and (3), $S_k = 0$ for $k \geq 2$. Thus $L(x) = S(x)$ and

$$(7) \quad \psi(x) = S\varphi(x) \quad \text{in } V.$$

It follows by identity theorem that (7) holds in U .

Example. Let X be a Banach space over the field of complex numbers $\mathbb{C}$ and let $E = P_1(X)$ be the Banach space of all continuous linear mappings of X into X .

Let ω be an invertible element of E such that $\|\omega\| < 1$, and $D := \sum_{n=0}^{\infty} \|\omega\|^n$. We put $G := \left\{ u \in E: \|u\| < \frac{1}{D} \right\}$ and

$$f: G \ni u \rightarrow f(u) = \omega u (I - u)^{-1} \in E,$$

where $I(x)=x$ for every $x \in X$. The above defined function f may be given by formula

$$f(u) = u + u^2 + \dots$$

and it is analytic in G (cf. [2] Th. 5.2), $A := f'(0) = \omega I_E$, $A^{-1}v = \omega^{-1}v$ for $v \in E$. The function

$$(8) \quad \varphi(u) := u[I - (I - \omega)^{-1}u]^{-1} = \sum_{n=1}^{\infty} u((I - \omega)^{-1}u)^{n-1}$$

is analytic solution of equation

$$\varphi[f(u)] = A\varphi(u)$$

in G . Indeed,

$$\begin{aligned} \varphi[f(u)] &= \omega u(I - u)^{-1} \{I - (I - \omega)^{-1} \omega u(I - u)^{-1}\}^{-1} = \\ &= \omega u(I - u)^{-1} \{(I - \omega)^{-1}[(I - \omega)(I - u) - \omega u](I - u)^{-1}\}^{-1} = \\ &= \omega u(I - u - \omega + \omega u - \omega u)^{-1}(I - \omega) = \omega u(I - u - \omega)^{-1}(I - \omega) \end{aligned}$$

and

$$\begin{aligned} A\varphi(u) &= \omega u[I - (I - \omega)^{-1}u]^{-1} = \omega u\{(I - \omega)^{-1}[(I - \omega) - u]\}^{-1} = \\ &= \omega u(I - \omega - u)^{-1}(I - \omega). \end{aligned}$$

For the mapping $\omega(x) = \omega(x_1, x_2) := \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have $\|\omega\| = \sup_{\|x\|=1} \|\omega(x)\| = \frac{1}{2}$, $\|\omega^{-1}\| = 4$ and $\|\omega\|^2 \|\omega^{-1}\| = 1$. Thus inequality (2) in hypothesis (II) is not necessary.

If $\|\omega\|^2 \|\omega^{-1}\| < 1$ then inequality (2) is fulfilled. In this case, we obtain formula (8) by application formula (6). This solution is invertible in a neighbourhood of zero by Theorem 2 and every analytic solution ψ of equation (1) in a neighbourhood of zero has the form $\psi(u) = S\varphi(u)$, where $S \in P_1(E)$ and $SA = AS$.

References

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