Categories of relations with standard factorizations

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§ 1. *Introduction.* In [10], D. PUPPE calls a category K an I-category ("Kategorie von Korrespondenzen" in [2] and [3], § 6.6, also called category of relations in [1], "catégorie ordonnée" in [6] or "catégorie à involution" in [7]) if the following conditions are fulfilled:

(a) For each pair of objects A, B, the set of morphisms $K(A, B)$ is partially ordered and the partial order is compatible with the composition, i.e. $f_1 \subset f_2 \Rightarrow \Rightarrow f_1 g \subset f_2 g$.

(b) There exists a contravariant functor *: $K \to K$ such that

(b1) $(fg)^{*} = g^{*} f^{*}$;

(b2) $f^{**} = f$;

(b3) $A^{*} = A$;

(b4) $f_1 \subset f_2 \Rightarrow f_1^{*} \subset f_2^{*}$.

D. Puppe has defined a relation from the object A to the object B of an abelian category $\mathscr{A}$ as a subobject of $A \oplus B$ and has given (§ 2 in [10]) a construction of the category $K(\mathscr{A})$ of relations over $\mathscr{A}$ ("die I-Kategorie $K(\mathscr{A})$ der Korrespondenzen über einer abelschen Kategorie $\mathscr{A}$") which satisfies the required (a) and (b) conditions. He has imposed a set of six axioms K1—K6 on a category of relations which give a characterization up to isomorphism of the category $K(\mathscr{A})$ of relations over $\mathscr{A}$.

In a previous paper [8], S. MACLANE considered a special sort of categories of relations, namely "partially ordered categories"; a partially ordered category $\mathscr{B}$ in the sense of MacLane is defined by three groups of axioms (I), (II), (III) (see [8]), from which we give here only the first one:

(I-a) To each morphism f: $A \to B$ there is a unique morphism f^{*}: $B \to A$ with $f^{**} = f = ff^{*} f$, $(fg)^{*} = g^{*} f^{*}$.

(I-b) Each set $\mathrm{Rel}\,(A, B)$ of morphisms f: $A \to B$ is a modular lattice under a partial order "$\subset$" such that, for f, g: $A \to B$, $g \subset f$ implies $g^{*} \subset f^{*}$, $gh \subset fh$.

The axioms (a), (b) and K1 $\to$ K5 due to D. Puppe, imply all axioms from MacLane [8] and are clearly *more restrictive than the latter,* according to [10] § 6 (we must exclude the condition of modularity for the lattice $\mathrm{Rel}\,(A, B)$ from I-b).

In the preceding paper [14] we have shown that a partially ordered category $\mathscr{B}$ can be embedded in a category of relations $\overline{\mathscr{B}}$ in which every map of $\mathscr{B}$ has a kernel and cokernel.

In our present paper we show that if we require besides axioms (I), (II), (III) of MacLane a natural condition of standard factorization of the morphisms, then we obtain a system equivalent to (a), (b) and K1→K5 from [10]. Moreover, this axiom of decomposition clearly valid for relations over an abelian category, allows us to cancel from (I-a) the equalities $f^{**}=f=ff^*f$, so that in the set of formal axioms such obtained these conditions of involution and regularity become theorems which we can prove ($f=ff^*f$ is regular in the sense of VON NEUMANN or "difonctionelle" according to RIGUET [11]).

We note that the morphisms of a category of relations which satisfy simultaneously $ff^*\subset 1$ and $f^*f\supset 1$ are called maps (in [1], "Abbildungen" in [2], "eigentliche Morphismen" in [10], "graphs" in [8]). The reason for this terminology can be seen clearly from the typical examples of categories of relations given by Puppe and Brinkmann (in [10], [1], [3]), RS, RG, RAb, RMod: the objects $A, B, C, \ldots$ are all sets, all groups, abelian groups, left modules over a fixed ring, respectively; the set of morphisms from A to B is the set of all relations $R\subset A\times B$ for RS and the set of all homomorphical relations $R\subset A\oplus B$ for RG, RAb, RMod.

Homomorphic relations have been first studied apparently by J. LAMBEK in [5], defined as subalgebras of the direct product $A\times B$ of two similar algebras (also [4]); submodules $R\subset A\oplus B$ of the direct product of two modules are called additive relations in [8] and [9], "Korrespondenzen" in [10] and [3] and linear relations in our paper [13]; homomorphic relations between vector spaces are called linear relations in [12].

We have mentioned above that the axioms (a), (b), K1—K6 due to Puppe characterize the category $K(\mathscr{A})$ of relations over an abelian category $\mathscr{A}$; let us recall here that if K is a category which satisfies only axioms (a), (b), K1—K3 (called pseudoexact category of relations, in [1] and [3]) then the subcategory of maps is merely exact and determines in the same way K up to isomorphism (we speak of exactness in the sense of MITCHELL, see [3], no addition is required as in the exact categories of BUCHSBAUM).

If we examine all the axioms imposed on a partially ordered category $\mathscr{B}$, the latter appears to satisfy essential properties of a bimodular category ("catégorie bimodulaire" [7]), all valid in the standard model RMod; but these axioms do not suffice to characterize the relations in an abelian category (Beispiel A in § 7 from [10]), so that it seems us natural to complete the system (I), (II), (III) in order to obtain a supplemented system equivalent to (a), (b) and K1→K6 (more precisely to (a), (b) and K1—K5).

§ 2. *Category of relations with standard factorizations.*

Definition. A category R is called a category of relations with standard factorizations if the following axioms are satisfied:

(I′) 1. Each $R(A, B)$ is a lattice under a partial order relation $\subset$ such that, for $f, g\in R(A, B)$, $g\subset f$ implies $gh\subset fh$ whenever the composites are defined.

2. There is a lattice isomorphism *: $R(A, B)\to R(B, A)$ for each pair of objects A, B such that $(fg)^*=g^*f^*$.

(IV) Every morphism $f\in R(A, B)$ may be expressed in the form $f=m_2e_2^*m_1^*e_1$

where m_i are faithful maps and e_i onto maps in R; in this decomposition of f each factor is uniquely determined up to isomorphism.

Remark. We say that a morphism $f \in R(A, B)$ is
a) universally-defined iff $f^\# f \supset 1_A$;
b) single-valued iff $ff^\# \subset 1_B$;
c) faithful iff $f^\# f \subset 1_A$;
d) onto iff $ff^\# \supset 1_B$.
f is called a map iff it is universally defined and single-valued.

Theorem 1. *In a category of relations with standard factorizations, $^\#$ is involutive ($f^{\#\#}=f$) and each morphism is regular in the sense of von Neumann.*

PROOF. Let A be an arbitrary object of R and 1_A the corresponding identity. Since $^\#: R(A, A) \to R(A, A)$ is a lattice isomorphism, there is a unique $f \in R(A, A)$ such that $f^\# = 1_A$. Then we have

$$1_A^\# 1_A = 1_A^\# f^\# = (f 1_A)^\# = f^\# = 1_A$$

and also

$$1_A^\# 1_A = 1_A^\#, \quad \text{hence} \quad 1_A^\# = 1_A.$$

Let now m be a faithful map in R, that is $m^\# m = 1$, $mm^\# \subset 1$. We prove that $m^{\#\#} = m$, by observing that $(m^\# m)^\# = 1^\#$ and then applying the above result $m^\# m^{\#\#} = 1$ which implies $mm^\# m^{\#\#} = m$, hence $m = (mm^\#)m^{\#\#} \subset m^{\#\#}$; conversely,

$$m^{\#\#} = m^{\#\#}(m^\# m) = (m^{\#\#} m^\#)m = (mm^\#)^\# m \subset 1^\# m = m.$$

Let next e be an onto map in R, that is $e^\# e \supset 1$, $ee^\# = 1$. The proof of the equality $e^{\#\#} = e$ is exactly dual to that of $m^{\#\#} = m$, in the sense that all inclusions are reversed and the order of composition is changed.

Finally, if f is any morphism of R which can be factored as $f = m_2 e_2^\# m_1^\# e_1$, then

$$f^{\#\#} = (m_2 e_2^\# m_1^\# e_1)^{\#\#} = (e_1^\# m_1^{\#\#} e_2^{\#\#} m_2^\#)^\# = (e_1^\# m_1 e_2 m_2^\#)^\# =$$
$$= m_2^{\#\#} e_2^\# m_1^\# e_1^{\#\#} = m_2 e_2^\# m_1^\# e_1 = f.$$

There is no difficulty in proving $ff^\# f = f$, as

$$ff^\# f = (m_2 e_2^\# m_1^\# e_1)(e_1^\# m_1 e_2 m_2^\#)(m_2 e_2^\# m_1^\# e_1) =$$
$$= (m_2 e_2^\# m_1^\#)(e_1 e_1^\#)(m_1 e_2)(m_2^\# m_2)(e_2^\# m_1^\# e_1) =$$
$$= (m_2 e_2^\# m_1^\#)(m_1 e_2)(e_2^\# m_1^\# e_1) = (m_2 e_2^\#)(m_1^\# e_1).$$

Corollary. In a category of relations with standard factorizations which satisfies also (II-a, b, c) and (III) from [8], all axioms due to MacLane are valid (again: we must exclude the condition of modularity from I-b).

Indeed, Theorem 1 shows that (I-a) is satisfied; obviously, (I-b) except the condition of modularity for the lattice $R(A, B)$, is contained in (I′). According to Puppe [10], § 6.1, the condition (II-d) from [8] can be proved using (I) and (II-a, b).

Remark. Clearly, the preceding corollary shows that the system of axioms (I′), (II), (III), (IV) is equivalent to (I), (II), (III), (IV) minus the condition of modularity from (I-b).

§ 3. *Categories of relations with standard factorizations which satisfy axiom (III) of MacLane.*

Let R be a category of relations with standard factorizations and denote by MR the subcategory of maps.

Lemma. *Suppose R satisfies also* (III) *from* [8]; *then there is a system of null morphisms in* MR.

Proof. Denote by ω_{AB} the least element from $R(A, B)$ for each pair of objects A, B of R; similarly, denote by Ω_{AB} the greatest element from $R(A, B)$ (there exist such morphisms since (III) holds).

For given objects A, B, C we put $O_{AB} = \omega_{CB}\Omega_{AC} \in R(A, B)$. Needless to say that O_{AB} depends only on A and B and not on C, as (III—2) is valid. O_{AB} is a map:

$$O_{AB}O^{\#}_{AB} = \omega\Omega\Omega^{\#}\omega^{\#} = \omega\Omega\Omega\omega = \omega\Omega\omega = \omega \subset 1.$$

$$O^{\#}_{AB}O_{AB} = \Omega^{\#}\omega^{\#}\omega\Omega = \Omega\omega\Omega = \Omega \supset 1.$$

For every $f \in MR(B, D)$ we have

$$fO_{AB} = f\omega_{CB}\Omega_{AC} \supset \omega_{BD}\omega_{CB}\Omega_{AC} = \omega_{CD}\Omega_{AC} = O_{AD}$$

and also

$$fO_{AB} = f\omega_{CB}\Omega_{AC} = f\omega_{DB}\omega_{CD}\Omega_{AC} \subset ff^{\#}\omega_{CD}\Omega_{AC} \subset 1_D\omega_{CD}\Omega_{AC} = O_{AD}.$$

Thus $fO_{AB} = O_{AD}$, as required. Dually, using $f^{\#}f \supset 1_E$ for all $f \in MR(E, A)$, we have $O_{AB}f = O_{EB}$.

Theorem 2. *Let R be a category of relations with standard factorizations which satisfies* (III).

a) *every map has a kernel and a cokernel in the subcategory MR.*

b) *$m \in MR(X, A)$ is a kernel in MR of the map $f \in MR(A, B)$ iff $\underline{m}$ is faithful and $mm^{\#} = 1_A \cap f^{\#}\omega_{BB}f$.*

c) *$e \in MR(B, Y)$ is a cokernel in MR of the map $f \in MR(A, B)$ iff $\underline{e}$ is onto and $e^{\#}e = 1_B \cup f\Omega_{AA}f^{\#}$.*

Proof. a$_1$) Let $s = 1_A \cap f^{\#}\omega_{BB}f \subset 1_A$ have the factorization $s = m_2e_2^{\#}m_1^{\#}e_1$. Consider $m = m_2e_2^{\#} \in R(X, A)$; the result will now follow from the fact that m is a faithful map with $mm^{\#} = s$.

Indeed, $mm^{\#} = ss^{\#} = s \subset 1_A$ and $m^{\#}m = (e_2m_2^{\#})(m_2e_2^{\#}) = e_2e_2^{\#} = 1_X$.

a$_2$) Now $m^{\#}m = 1_X$, hence m is a monomorphism in R; we have also

$$fm = f(mm^{\#}m) = f(mm^{\#})m = f(1_A \cap f^{\#}\omega_{BB}f)m \subset f(f^{\#}\omega_{BB}f)m =$$

$$= (ff^{\#})\omega_{BB}fm \subset \omega_{BB}fm \subset \omega_{BB}\Omega_{AB}\Omega_{XA} = \omega_{BB}\Omega_{XB} = O_{XB},$$

hence $fm = O_{XB}$ (if $f_1 \subset f_2$ are maps, then $f_1 = f_2$!).

Finally, we show that $fn=O_{YB}$ for $n\in MR(Y,A)$ implies $n=mx$ for some $x\in MR(Y,X)$. Take $x=m^{\#}n$; $fn=O_{YB}$ implies

$$nn^{\#}=(f^{\#}f)nn^{\#}=f^{\#}O_{YB}n^{\#}=f^{\#}\omega_{BB}(\omega_{BB}\Omega_{YB})n^{\#}=f^{\#}\omega_{BB}(fn)n^{\#}=$$
$$=(f^{\#}\omega_{BB}f)(nn^{\#})\subset f^{\#}\omega_{BB}f;$$

then

$$x^{\#}x=(m^{\#}n)^{\#}(m^{\#}n)=n^{\#}(mm^{\#})n=n^{\#}(1_A\cap f^{\#}\omega_{BB}f)n\supset n^{\#}(nn^{\#})n=$$
$$=n^{\#}(nn^{\#}n)=n^{\#}n\supset 1_Y$$

and

$$xx^{\#}=(m^{\#}n)(m^{\#}n)^{\#}=(m^{\#}n)(n^{\#}m)\subset m^{\#}m=1_X,$$

so we conclude that x is a map; it remains to note that

$$mx=mm^{\#}n\subset 1_An=n \quad \text{implies} \quad mx=n.$$

a$_3$) Dually, one can prove that if $q=1_B\cup f\Omega_{BB}f^{\#}=m_2e_2^{\#}m_1^{\#}e_1$, then $e=e_2m_2^{\#}$ is an onto map and it is a cokernel im MR for f.

b) *By* a_2) if m is a faithful map with $mm^{\#}=1_A\cap f^{\#}\omega_{BB}f$, then m is a kernel of f in MR. Conversely, if n is a kernel of f in MR, then by a) there exist a kernel m of f with $mm^{\#}=1_A\cap f^{\#}\omega_{BB}f$ and we have necessarily $n=mi$, where i is an isomorphism. Now $ii^{-1}=1$, $i^{-1}i=1$ imply

$$i^{\#}i=(i^{-1}i)i^{\#}i=i^{-1}(ii^{\#}i)=i^{-1}i=1$$

and

$$ii^{\#}=ii^{\#}(ii^{-1})=(ii^{\#}i)i^{-1}=ii^{-1}=1,$$

i.e. i is a faithful and onto map. Therefore $n=mi$ as the composite of two faithful maps is also a faithful map and

$$nn^{\#}=mi(mi)^{\#}=mii^{\#}m=mm^{\#}=1_A\cap f^{\#}\omega_{BB}f.$$

c) The proof is dual to that of b).

Corollary. *If a non-empty category R of relations with standard factorizations satisfies* (III) *then* K1 *and* K4 *from* [10] *are valid.*

Indeed $1_A\in R(A,A)$ is an identity and $m\in MR(X,A)$ its kernel in MR, so X is a null object in MR; moreover, (III) assures that X satisfies all conditions from K1. Clearly, K4 is contained in (I′).

§ 4. **Theorem 3.** *In a non-empty category of relations with standard factorizations which satisfies also the conditions* (II-a, b, c) *and* (III) *from* [8], *axioms* K1, K2, K3, K4, K5 *due to Puppe are all valid.*

Proof. Axioms K1 and K4 are fulfilled according to the corollary of Theorem 2.

Axiom K2 is also valid; by the corollary of Theorem 1 all conditions (I), (II), (III) of Mac Lane are fulfilled, so that one can prove the validity of K2 (see [10], § 6, 3, this proof makes use esentially of II-c!).

The proof of K3-a; for $u\in R(O,A)$ one can construct, just as in a$_1$) from the proof of Theorem 2, a faithful map $m\in MR(X,A)$ with $mm^{\#}=s=uu^{\#}\cap 1_A\subset 1_A$; then

$$m\Omega_{OX}\supset mm^{\#}u=(uu^{\#}\cap 1_A)u=(\text{II-a})u$$

and

$$m\Omega_{OX}=mm^{\#}m\Omega_{OX}\subset uu^{\#}m\Omega_{OX}=u1_O=u, \quad \text{so that} \quad m\Omega_{OX}=u.$$

The proof of K3-b: for $u \in R(O, A)$ one can construct, as in a_3) from the proof of Theorem 2, an onto map $e \in MR(A, Y)$ with $e^{\#} e = q = 1_B \cup uu^{\#} \supset 1_B$; then

$$e^{\#} \omega_{OY} \subset e^{\#} eu = (1_B \cup uu^{\#})u = (\text{II-b})u$$

and

$$e^{\#} \omega_{OY} = e^{\#} ee^{\#} \omega_{OY} \supset uu^{\#} e^{\#} \omega_{OY} = u 1_O = u, \quad \text{so that} \quad e^{\#} \omega_{OY} = u.$$

For axiom K5; if K1, K2, K3, K4 are fulfilled K5 is equivalent to (II-a)+(II-b) (see [10], § 6.10).

Corollary. *The system of axioms* (I′), (II), (III), (IV) *is equivalent to conditions* (a), (b) *from the Introduction together with axioms* K1→K5 *of* D. PUPPE.

PROOF. Clearly, by Theorem 3, (I′), (II-a, b, c), (III), (IV) imply (a), (b), K1→K5.

Conversely, (a), (b), K1→K5⇒(I-a, b), (II-a, b, c), (III) according to Puppe and (I-a, b)⇒(I′).

Theorem 4.7 from [10] shows that if we have (a), (b), K1, K2, K3, then every morphism f can be factored as $f = m_1^{\#} e_2 m_2 e_1^{\#}$, where m_i are I—D—K regular and e_i are I—D—B regular; but I—D—K regularity is equivalent to the definition of faithful maps and I—D—B regularity is equivalent to the definition of onto maps (1.18 in [10]); hence if we take the factorization of $f^{\#}$ in the form $f^{\#} = \bar{m}_1^{\#} \bar{e}_2 \bar{m}_2 \bar{e}_1^{\#}$, we obtain $f = \bar{e}_1 \bar{m}_2^{\#} \bar{e}_2^{\#} \bar{m}_1$ as in axiom (IV).

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