

Existence and uniqueness of solutions of a functional-differential equation

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1. Introduction

In the present paper we are concerned with the functional-differential equation of the form

$$(1) \quad \varphi'(x) = h(x, \varphi(x), \varphi[f(x, \varphi(x))], u)$$

with initial condition

$$(2) \quad \varphi(0) = z_0$$

where φ is an unknown function and h, f are known functions and u is a real parameter.

We shall prove that the problem (1)–(2) has exactly one solution defined in the interval $(0, \infty)$ and belonging to a certain function class G , which is defined below and this solution depends continuously on u .

For the equation

$$\varphi'(x) = h(x, \varphi(x), \varphi[f_1(x)], \dots, \varphi[f_n(x)], u)$$

the corresponding problem has been investigated by author in [3]. The problem of the local existence of solutions of equation (1) has been investigated in [4].

2. Existence and uniqueness

In this section we are going to establish a theorem on the existence of a unique solution of the initial-value problem (1)–(2).

We assume the following

HYPOTHESIS 1.

(i) Let $(Y, \|\cdot\|)$ be a Banach space. The functions $h: I \times Y^2 \times R \rightarrow Y, f: I \times Y \rightarrow I$ where $I = (0, \infty), R = (-\infty, +\infty)$ are continuous on $I \times Y^2 \times R$ and $I \times Y$ respectively.

(ii) There exist continuous functions $L_i: I \rightarrow I, i=1, 2$ such that for every $z_i, y_i \in Y, i=1, 2, x \in I$ and $u \in R$ we have

$$\|h(x, z_1, z_2, u) - h(x, y_1, y_2, u)\| \leq L_1(x)\|z_1 - y_1\| + L_2(x)\|z_2 - y_2\|.$$

(iii) There exist constants $F \geq 0$, $r > 1$, $\beta \geq 0$ such that for every $y, y_1 \in Y$, $x \in I$

$$|f(x, y) - f(x, y_1)| \leq F \exp(-rL(x)) \|y - y_1\|$$

where

$$(3) \quad L(x) = \int_0^x [L_1(s) + (\beta F + 1)L_2(s)] ds, \quad x \geq 0.$$

(iv) There exist nonnegative constants A, B, C such that for every $y_1, y_2 \in Y$, $x \in I$ and $u \in R$

$$\|h(x, y_1, y_2, u)\| \leq A \exp(rL(x)) + B\|y_1\| + C\|y_2\|.$$

(v) There exists constant $N \geq 0$ such that for $x \in I$, $u \in R$

$$\int_0^x \|h(s, 0, 0, u)\| ds \leq N \exp(rL(x)).$$

Let X be the space of all functions $\varphi: I \rightarrow Y$ which are continuous in I and

$$(4) \quad \sup_{x \in I} (\|\varphi(x)\| \exp(-rL(x))) < \infty.$$

X with the norm (cf. [1])

$$(5) \quad \|\varphi\| = \sup_{x \in I} (\|\varphi(x)\| \exp(-rL(x)))$$

is the Banach space.

We define G as the space of these functions $\varphi \in X$ which fulfil the inequalities

$$(6) \quad \|\varphi(x)\| \leq \alpha \exp(rL(x)), \quad x \in I,$$

$$(7) \quad \|\varphi(x) - \varphi(z)\| \leq \beta \exp(rL(x)) |x - z|, \quad x, z \in I, \quad x \geq z, \quad \alpha, \beta - \text{constants}.$$

We can verify that G with metric

$$(8) \quad d(\varphi, \psi) = \|\varphi - \psi\|$$

is a complete metric space.

Now we shall prove

Theorem 1. Suppose that hypothesis 1 is fulfilled and let

$$(9) \quad f(x, y) \equiv x, \quad x \in I, \quad y \in Y.$$

If, moreover, the numbers α, β fulfil the inequalities

$$(10) \quad \alpha \geq (N + \|z_0\|)r(r-1)^{-1},$$

$$(11) \quad \beta \geq A + B\alpha + C\alpha$$

then, for every $u \in R$, the initial-value problem (1)–(2) ($z_0 \in Y$) has exactly one solution $\varphi \in G$, given as the limit of successive approximations.

PROOF. Let $u \in R$ be fixed. Equation (1) with initial condition (2) is equivalent with the equation

$$\varphi(x) = z_0 + \int_0^x h(s, \varphi(s), \varphi[f(s, \varphi(s))], u) ds.$$

Now we define the transformation $\Phi = T\varphi$ by the formula

$$(12) \quad \Phi(x) = z_0 + \int_0^x h(s, \varphi(s), \varphi[f(s, \varphi(s))], u) ds.$$

We shall prove that (12) transforms G into itself. Let $\varphi \in G$. In view of (i) Φ is continuous in I . From (ii), (6), (9), (3), (v) and (10) we obtain

$$\begin{aligned} \|\Phi(x)\| &\leq \int_0^x \|h(s, \varphi(s), \varphi[f(s, \varphi(s))], u) - h(s, 0, 0, u)\| ds + \\ &\quad + \int_0^x \|h(s, 0, 0, u)\| ds + \|z_0\| \leq \\ &\leq \int_0^x \{L_1(s)\|\varphi(s)\| + L_2(s)\|\varphi[f(s, \varphi(s))]\| \} ds + N \exp(rL(x)) + \|z_0\| \leq \\ &\leq \int_0^x \{L_1(s)\alpha \exp(rL(s)) + L_2(s)\alpha \exp(rL[f(s, \varphi(s))])\} ds + N \exp(rL(x)) + \|z_0\| \leq \\ &\leq \alpha \int_0^x \{L_1(s) + L_2(s)\} \exp(rL(s)) ds + N \exp(rL(x)) + \|z_0\| \leq \\ &\leq \frac{\alpha}{r} \int_0^x r \{L_1(s) + (\beta F + 1)L_2(s)\} \exp(rL(s)) ds + N \exp(rL(x)) + \|z_0\| \leq \\ &\leq \frac{\alpha}{r} [\exp(rL(x)) - 1] + N \exp(rL(x)) + \|z_0\| \leq \alpha \exp(rL(x)). \end{aligned}$$

Hence Φ fulfils (6). Next, from (iv), (6), (9), (11) for $x \geq z$ we have

$$\begin{aligned} \|\Phi(x) - \Phi(z)\| &\leq \left\| \int_0^x h(s, \varphi(s), \varphi[f(s, \varphi(s))], u) ds \right\| \leq \\ &\leq \int_z^x (A + B\alpha + C\alpha) \exp(rL(s)) ds \leq \exp(rL(x)) \int_z^x (A + B\alpha + C\alpha) ds \leq \\ &\leq (A + B\alpha + C\alpha) \exp(rL(x)) |x - z| \leq \beta \exp(rL(x)) |x - z| \end{aligned}$$

and condition (7) is fulfilled.

Now we shall prove that the transformation (12) is a contraction map. Actually, from (ii), (8), (7), (iii), (9), (3) for $\Phi = T\varphi$, $\Psi = T\psi$, where $\varphi, \psi \in G$, we get

$$\begin{aligned}
 & \|\Phi(x) - \Psi(x)\| \cong \\
 & \cong \int_0^x \|h(s, \varphi(s), \varphi[f(s, \varphi(s))], u) - h(s, \psi(s), \psi[f(s, \psi(s))], u)\| ds \cong \\
 & \cong \int_0^x \{L_1(s) \|\varphi(s) - \psi(s)\| + L_2(s) \|\varphi[f(s, \varphi(s))] - \psi[f(s, \psi(s))]\| \} ds \cong \\
 & \cong \int_0^x \{L_1(s) \|\varphi(s) - \psi(s)\| + L_2(s) (\|\varphi[f(s, \varphi(s))] - \varphi[f(s, \psi(s))]\| + \\
 & \quad + \|\varphi[f(s, \psi(s))] - \psi[f(s, \psi(s))]\|) \} ds \cong \\
 & \cong \int_0^x \{L_1(s) d(\varphi, \psi) \exp(rL(s)) + L_2(s)(\beta F + 1) d(\varphi, \psi) \exp(rL(s))\} ds = \\
 & = \frac{1}{r} d(\varphi, \psi) \int_0^x r \{L_1(s) + (\beta F + 1) L_2(s)\} \exp(rL(s)) ds = \\
 & = \frac{1}{r} d(\varphi, \psi) [\exp(rL(x)) - 1].
 \end{aligned}$$

Hence

$$d(\Phi, \Psi) \cong \frac{1}{r} d(\varphi, \psi).$$

On account of Banach's fixed-point theorem there exist exactly one solution $\varphi \in G$ of the problem (1)–(2), which completes the proof.

3. Continuous dependence on parameter

Now we consider the problem of the continuous dependence of solutions of the problem (1)–(2) on parameter u .

We assume the following

HYPOTHESIS 2.

There exist constant M and functions $P: I \rightarrow I$, $\omega: I \rightarrow I$ such that $\omega(u) \rightarrow 0$ as $u \rightarrow 0+$ and

$$\exp(-rL(x)) \int_0^x P(s) ds \cong M, \quad x \in I.$$

Moreover, for $z_1, z_2 \in Y$, $x \in I$, $u_1, u_2 \in R$ we have

$$\|h(x, z_1, z_2, u_1) - h(x, z_1, z_2, u_2)\| \cong P(x) \omega(|u_1 - u_2|).$$

We have the following

Theorem 2. *Suppose that hypotheses of theorem 1 and hypothesis 2 are fulfilled. Then solutions of the problem (1)—(2) depends continuously on u .*

PROOF. For $\varphi \in G$ we define

$$F(x, \varphi, u) = z_0 + \int_0^x h(s, \varphi(s), \varphi[f(s, \varphi(s))], u) ds.$$

Similarly as in the proof of theorem 1, for $\varphi, \psi \in G$ we obtain

$$(13) \quad d[F(x, \varphi, u), F(x, \psi, u)] \leq \frac{1}{r} d(\varphi, \psi).$$

In view of hypothesis 2 we also have

$$(14) \quad d[F(x, \varphi, u_1), F(x, \varphi, u_2)] \leq M\omega(|u_1 - u_2|).$$

From (13), (14) and applying the Banach's fixed-point principle ([2], theorem 3.1, p. 18) we obtain our assertion.

References

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