

Curvature theory of generalized second order gauge connections

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Abstract. Lately a big attention has been paid to second order gauge connections, but the investigation are mostly restricted to the d -connection. Here this connection is generalized. The curvature tensors and the Ricci equations are obtained, which for the generalized connection have a rather simple form.

1. Adapted basis in TF

Let F be an $n + m + l$ dimensional C^∞ manifold. Some point $v \in F$ in the local charts (U, φ) and (U', φ') has coordinates (x^i, y^a, z^p) and $(x^{i'}, y^{a'}, z^{p'})$ respectively. In $U \cap U'$ the allowable coordinate transformations are given by the equations

$$(1.1) \quad \begin{aligned} x^{i'} &= x^{i'}(x) & i, j, h, k &= \overline{1, n}, \\ y^{a'} &= y^{a'}(x, y) & a, b, c, d, e &= \overline{n+1, n+m}, \\ z^{p'} &= z^{p'}(x, z) & p, q, r, s, t &= \overline{n+m+1, n+m+l}, \end{aligned}$$

where

$$(1.2) \quad \text{rank} \left[\frac{\partial x^{i'}}{\partial x^i} \right] = n, \text{rank} \left[\frac{\partial y^{a'}}{\partial y^a} \right] = m, \text{rank} \left[\frac{\partial z^{p'}}{\partial z^p} \right] = l.$$

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From (1.1) and (1.2) it follows that inverse transformations of the form

$$(1.1)' \quad x^i = x^i(x'), \quad y^a = y^a(x', y'), \quad z^p = z^p(x', z')$$

exist.

Proposition 1.1. *The coordinate transformations of type (1.1) form a group.*

The tangent space TF is spanned at any point $v \in F$ by $n + m + l$ basis vectors, which form the basis

$$(1.3) \quad \bar{B} = \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial z^p} \right\}.$$

The elements of $\bar{B}$ with respect to (1.1) are transformed in the following way:

$$(1.4) \quad \frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial}{\partial x^{i'}} + \frac{\partial y^{a'}}{\partial x^i} \frac{\partial}{\partial y^{a'}} + \frac{\partial z^{p'}}{\partial x^i} \frac{\partial}{\partial z^{p'}},$$

$$(1.5) \quad \frac{\partial}{\partial y^a} = \frac{\partial y^{a'}}{\partial y^a} \frac{\partial}{\partial y^{a'}}, \quad \frac{\partial}{\partial z^p} = \frac{\partial z^{p'}}{\partial z^p} \frac{\partial}{\partial z^{p'}}.$$

The functions $\mathcal{N}_{i'}^{b'}(x', y')$ and $\mathcal{M}_{i'}^{p'}(x', z')$ are the coefficients of non-linear connections of the second order if they satisfy the following law of transformation:

$$(1.6) \quad \begin{aligned} \text{(a)} \quad \mathcal{N}_i^b(x, y) &= \mathcal{N}_{i'}^{b'}(x', y') \frac{\partial x^{i'}}{\partial x^i} \frac{\partial y^b}{\partial y^{b'}} + \frac{\partial y^{a'}}{\partial x^i} \frac{\partial y^b}{\partial y^{a'}}, \\ \text{(b)} \quad \mathcal{M}_i^p(x, z) &= \mathcal{M}_{i'}^{p'}(x', z') \frac{\partial x^{i'}}{\partial x^i} \frac{\partial z^p}{\partial z^{p'}} + \frac{\partial z^{p'}}{\partial x^i} \frac{\partial z^p}{\partial z^{p'}}. \end{aligned}$$

Remark. In many papers as in [11], [13], [14] $\mathcal{N}$ and $\mathcal{M}$ are interchanged. The reason for this change is the fact that for $n + m$ dimensional F , $\mathcal{N}$ is reduced to N used before as in [6]–[10].

Using the notation

$$(1.7) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{N}_i^b(x, y) \frac{\partial}{\partial y^b} - \mathcal{M}_i^p(x, z) \frac{\partial}{\partial z^p}$$

from (1.6) and (1.7) we obtain

$$(1.8) \quad \frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i} \frac{\delta}{\delta x^{i'}}.$$

Thus the adapted basis

$$(1.9) \quad B(\mathcal{N}, \mathcal{M}) = \left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial z^p} \right\}$$

of TF is obtained. It is clear that as many adapted bases can be formed, as functions $\mathcal{N}$ and $\mathcal{M}$ satisfying (1.6)(a) and (1.6)(b) can be found. The transformations of the elements of $B(\mathcal{N}, \mathcal{M})$, $T_H F$, $T_{V_1} F$, $T_{V_2} F$ are determined by (1.5) and (1.8).

Let us denote by $T_H F$, $T_{V_1} F$, $T_{V_2} F$ the subspaces of TF spanned by $\{\frac{\delta}{\delta x^i}\}$, $\{\frac{\partial}{\partial y^a}\}$, $\{\frac{\partial}{\partial z^p}\}$ respectively, then

$$(1.10) \quad TF = T_H F \oplus T_{V_1} F \oplus T_{V_2} F.$$

Any vector field X in TF can be written in the form

$$(1.11) \quad X = X^i \frac{\delta}{\delta x^i} + X^a \frac{\partial}{\partial y^a} + X^p \frac{\partial}{\partial z^p}.$$

The coordinates of X under (1.1) are transformed in the following way:

$$(1.12) \quad X^{i'} = X^i \frac{\partial x^{i'}}{\partial x^i}, \quad X^{a'} = X^a \frac{\partial y^{a'}}{\partial y^a}, \quad X^{p'} = X^p \frac{\partial z^{p'}}{\partial z^p}.$$

The adapted basis of T^*F is

$$(1.13) \quad B^*(\mathcal{N}, \mathcal{M}) = \{dx^i, \delta y^a, \delta z^p\},$$

where

$$(1.14) \quad \delta y^a = dy^a + \mathcal{N}_i^a(x, y) dx^i$$

$$(1.15) \quad \delta z^p = dz^p + \mathcal{M}_i^p(x, z) dx^i.$$

The elements of B^* are transformed in the following way:

$$(1.16) \quad dx^i = \frac{\partial x^i}{\partial x^{i'}} dx^{i'}, \quad \delta y^a = \frac{\partial y^a}{\partial y^{a'}} \delta y^{a'}, \quad \delta z^p = \frac{\partial z^p}{\partial z^{p'}} \delta z^{p'}.$$

Let us denote the subspaces of T^*F spanned by $\{dx^i\}$, $\{\delta y^a\}$, $\{\delta z^p\}$ respectively by $T_H^* F$, $T_{V_1}^* F$, $T_{V_2}^* F$, then

$$(1.17) \quad T^*F = T_H^* F \oplus T_{V_1}^* F \oplus T_{V_2}^* F.$$

Any 1-form field $\omega \in T^*F$ can be written in the form:

$$(1.18) \quad \omega = \omega_i dx^i + \omega_a \delta y^a + \omega_p \delta z^p.$$

The coordinates of the 1-form ω with respect to (1.1) are transformed in the following way:

$$(1.19) \quad \omega_i = \omega_{i'} \frac{\partial x^{i'}}{\partial x^i}, \quad \omega_a = \omega_{a'} \frac{\partial y^{a'}}{\partial y^a}, \quad \omega_p = \omega_{p'} \frac{\partial z^{p'}}{\partial z^p}.$$

2. Gauge covariant derivatives of the second order

Let $\nabla : TF \times TF \rightarrow TF$ ($\times$ is the Descartes product) be a linear connection, such that $\nabla : (X, Y) \rightarrow \nabla_X Y \in TF$, $\forall X, Y \in TF$. The operator ∇ is called generalized gauge connection of the second order. It is called d -gauge connection of the second order if $\nabla_X Y$ is in $T_H F$, $T_{V_1} F$ or $T_{V_2} F$ provided Y is in $T_H F$, $T_{V_1} F$ or $T_{V_2} F$ respectively, $\forall X \in TF$. It has been studied by many authors, mostly romanian geometers.

Here we shall not make this restriction on ∇ . In the following we shall use the abbreviations: $\delta_k = \frac{\delta}{\delta x^k}$, $\partial_k = \frac{\partial}{\partial x^k}$, $\partial_a = \frac{\partial}{\partial y^a}$, $\partial_p = \frac{\partial}{\partial z^p}$.

Definition 2.1. The generalized gauge connection ∇ of the second order is defined by

$$(2.1) \quad \begin{aligned} (a) \quad & \nabla_{\delta_i} \delta_\beta = F_{\beta i}^\kappa \delta_\kappa, \\ (b) \quad & \nabla_{\partial_a} \delta_\beta = C_{\beta a}^\kappa \delta_\kappa \\ (c) \quad & \nabla_{\partial_p} \delta_\beta = L_{\beta p}^\kappa \delta_\kappa, \end{aligned}$$

where $\beta = j$ or $\beta = b$ or $\beta = q$ and

$$(2.2) \quad T_{\dots}^\kappa \delta_\kappa = T_{\dots}^{\dots k} \delta_k + T_{\dots}^{\dots c} \partial_c + T_{\dots}^{\dots r} \partial_r.$$

We shall use the abbreviated form of (2.1):

$$(2.3) \quad \nabla_{\delta_\alpha} \delta_\beta = \Gamma_{\beta \alpha}^\kappa \delta_\kappa.$$

From (2.2) and (2.3) it follows:

If $\alpha = i$, then $\Gamma = F$; if $\alpha = a$, then $\Gamma = C$; if $\alpha = p$, then $\Gamma = L$.

Proposition 2.1. *If X is the vector field (1.11) defined on F , then the following equations are valid:*

$$(2.4) \quad \begin{aligned} \nabla_{\delta_i} X &= X|_i \delta_\alpha, & X|_i &= \delta_1 X^\alpha + F_{\beta i}^\alpha X^\beta \\ \nabla_{\partial_a} X &= X^\alpha|_a \delta_\alpha, & X^\alpha|_a &= \partial_a X^\alpha + C_{\beta a}^\alpha X^\beta \\ \nabla_{\partial_p} X &= X^\alpha|_p \delta_\alpha, & X^\alpha|_p &= \partial_p X^\alpha + L_{\beta p}^\alpha X^\beta, \end{aligned}$$

where

$$(2.5) \quad \Gamma_{\dot{\beta}} X^{\beta} = \Gamma_{\dot{j}} X^j + \Gamma_{\dot{b}} X^b + \Gamma_{\dot{q}} X^q.$$

Theorem 2.1. *If X and Y are vector fields in TF expressed in the basis B , and ∇ the second order gauge connection defined by (2.1), then the following equation is valid:*

$$(2.6) \quad \nabla_Y X = (X^\alpha_{|\beta}) Y^\beta \delta_\alpha,$$

where

$$(2.7) \quad \dots_{|\beta} Y^\beta = \dots_{|j} Y^j + \dots_{|b} Y^b + \dots_{|q} Y^q.$$

Theorem 2.2. *All covariant derivatives $X^\alpha_{|i}$, $X^\alpha_{|a}$, $X^\alpha_{||p}$ ($\alpha = j$, or $\alpha = b$, or $\alpha = p$) from (2.4) are transformed as tensors with respect to (1.1) if all connection coefficients from (2.1) are transformed as tensors, except the following which have the form:*

$$(2.8) \quad F_j^k{}^i = F_{j' i'}^{k'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^{j'}}{\partial x^j} + \frac{\partial^2 x^{k'}}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial x^{k'}}$$

$$(2.9) \quad F_b^c{}^i = F_{b' i'}^{c'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial y^{b'}}{\partial y^b} \frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial x^i \partial y^b} \frac{\partial y^c}{\partial y^{c'}} - \mathcal{N}_i^a \frac{\partial^2 y^{c'}}{\partial y^b \partial y^a} \frac{\partial y^c}{\partial y^{c'}}$$

$$(2.10) \quad F_q^r{}^i = F_{q' i'}^{r'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial z^{q'}}{\partial z^q} \frac{\partial z^r}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial x^i \partial z^q} \frac{\partial z^r}{\partial z^{r'}} - \mathcal{M}_i^s \frac{\partial^2 z^{r'}}{\partial z^s \partial z^q} \frac{\partial z^r}{\partial z^{r'}}$$

$$(2.11) \quad C_b^c{}^a = C_{b' a'}^{c'} \frac{\partial y^{b'}}{\partial y^b} \frac{\partial y^{a'}}{\partial y^a} \frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial y^a \partial y^b} \frac{\partial y^c}{\partial y^{c'}}$$

$$(2.12) \quad L_q^r{}^p = L_{q' p'}^{r'} \frac{\partial z^{q'}}{\partial z^q} \frac{\partial z^{p'}}{\partial z^p} \frac{\partial z^r}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial z^q \partial z^p} \frac{\partial z^r}{\partial z^{r'}}$$

The proof is given in [7]

Theorem 2.3. *The torsion tensor T for the second order gauge connection ∇ has the form:*

$$(2.13) \quad T(X, Y) = T_{\alpha\beta}^{\kappa} Y^\alpha X^\beta \delta_\kappa,$$

where

$$(2.14) \quad T_{\alpha\beta}^{\kappa} = \Gamma_{\alpha\beta}^{\kappa} - \Gamma_{\beta\alpha}^{\kappa}$$

except the following components:

$$\begin{aligned}
 (2.15) \quad & (a) \quad T_{ji}^c = F_{ji}^c - F_{ij}^c - \mathcal{N}_{ij}^c \\
 & (b) \quad T_{jb}^c = C_{jb}^c - F_{bj}^c + \partial_b \mathcal{N}_j^c = -T_{bj}^c \\
 & (c) \quad T_{ji}^r = F_{ji}^r - F_{ij}^r - \mathcal{M}_{ij}^q \\
 & (d) \quad T_{pi}^r = F_{pi}^r - L_{ip}^r - \partial_p \mathcal{M}_i^r = -T_{pi}^r,
 \end{aligned}$$

where

$$\begin{aligned}
 (2.16) \quad & (a) \quad \mathcal{N}_{ij}^c = (\partial_j \mathcal{N}_i^c - \mathcal{N}_j^b \partial_b \mathcal{N}_i^c) - (i, j) \\
 & (b) \quad \mathcal{M}_{ij}^q = (\partial_j \mathcal{M}_i^q - \mathcal{M}_j^p \partial_p \mathcal{M}_i^q) - (i, j)
 \end{aligned}$$

The proof is given in [7].

3. The curvature theory of ∇

The curvature tensor for the second order gauge connection ∇ is defined as usual

$$(3.1) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

If we use the notation

$$X = X^\alpha \partial_\alpha, \quad Y = Y^\beta \partial_\beta, \quad Z = Z^\gamma \partial_\gamma,$$

then we have

$$\begin{aligned}
 (3.2) \quad & \nabla_X \nabla_Y Z = \nabla_{X^\alpha \partial_\alpha} \nabla_{Y^\beta \partial_\beta} Z^\gamma \partial_\gamma \\
 & = \nabla_{X^\alpha \partial_\alpha} [Y^\beta (\partial_\beta Z^\gamma) \partial_\gamma + Y^\beta Z^\gamma \Gamma_{\gamma\beta}^\kappa \partial_\kappa] \\
 & = X^\alpha (\partial_\alpha Y^\beta) (\partial_\beta Z^\gamma) \partial_\gamma + X^\alpha Y^\beta (\partial_\alpha \partial_\beta Z^\gamma) \partial_\gamma \\
 & \quad + X^\alpha Y^\beta (\partial_\beta Z^\gamma) \Gamma_{\gamma\alpha}^\kappa \partial_\kappa + X^\alpha (\partial_\alpha Y^\beta) Z^\gamma \Gamma_{\gamma\beta}^\kappa \partial_\kappa \\
 & \quad + X^\alpha Y^\beta (\partial_\alpha Z^\gamma) \Gamma_{\gamma\beta}^\kappa \partial_\kappa + X^\alpha Y^\beta Z^\gamma (\partial_\alpha \Gamma_{\gamma\beta}^\kappa) \partial_\kappa \\
 & \quad + X^\alpha Y^\beta Z^\gamma \Gamma_{\gamma\beta}^\theta \Gamma_{\theta\alpha}^\kappa \partial_\kappa.
 \end{aligned}$$

From

$$(3.3) \quad [X, Y] = X^\alpha (\partial_\alpha Y^\beta) \partial_\beta - Y^\alpha (\partial_\alpha X^\beta) \partial_\beta + X^\alpha Y^\beta (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha),$$

where

$$\begin{aligned}
 (3.4) \quad & X^\alpha Y^\beta (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) = X^\alpha Y^\beta K_{\alpha\beta}^\kappa \partial_\kappa \\
 & = X^i Y^j (\mathcal{N}_{ij}^c \partial_c + \mathcal{M}_{ij}^q \partial_q) + (X^i Y^b - Y^i X^b) (\partial_b \mathcal{N}_i^c) \partial_c \\
 & \quad + (X^i Y^q - Y^i X^q) (\partial_q \mathcal{M}_i^p) \partial_p,
 \end{aligned}$$

it follows:

$$\begin{aligned}
 \nabla_{[XY]}Z &= X^\alpha(\partial_\alpha Y^\beta)(\partial_\beta Z^\gamma)\partial_\gamma + X^\alpha(\partial_\alpha Y^\beta)Z^\gamma\Gamma_{\gamma\alpha}^\kappa\partial_\kappa \\
 &\quad - Y^\alpha(\partial_\alpha X^\beta)(\partial_\beta Z^\gamma)\partial_\gamma - Y^\alpha(\partial_\alpha X^\beta)Z^\gamma\Gamma_{\gamma\beta}^\kappa\partial_\kappa \\
 &\quad + X^\alpha Y^\beta[(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)Z^\gamma]\partial_\gamma \\
 &\quad + X^\alpha Y^\beta Z^\gamma K_{\alpha\beta}^\theta\Gamma_{\gamma\theta}^\kappa\partial_\kappa.
 \end{aligned}
 \tag{3.5}$$

Substituting (3.2) and (3.5) into (3.1) we obtain

$$\begin{aligned}
 R(X, Y)Z &= [K_{\gamma\beta\alpha}^\kappa X^\alpha Y^\beta - (\mathcal{N}_{ij}^c C_{\gamma c}^\kappa + \mathcal{M}_{ij}^q L_{\gamma q}^\kappa)X^i Y^j \\
 &\quad - \partial_b \mathcal{N}_i^c C_{\gamma c}^\kappa (X^i Y^b - Y^i X^b) \\
 &\quad - \partial_q \mathcal{M}_i^p L_{\gamma p}^\kappa (X^i Y^q - Y^i X^q)]Z^\gamma \partial_\kappa,
 \end{aligned}
 \tag{3.6}$$

where

$$K_{\gamma\beta\alpha}^\kappa = (\partial_\alpha \Gamma_{\gamma\beta}^\kappa - \Gamma_{\gamma\alpha}^\theta \Gamma_{\theta\beta}^\kappa) - (\alpha, \beta).$$

As the indices $\alpha, \beta, \gamma, \kappa$ belong to one of the sets $\{i, j, k, l, \dots\}$, $\{a, b, c, d, \dots\}$, $\{p, q, r, s, t, \dots\}$ (corresponding to $T_H F$, $T_{V_1} F$, $T_{V_2} F$ respectively), so on the TF we have $3^4 = 81$ types of curvature tensors. It is meaningless to introduce different letters as R, P, S for the curvature tensors as in Finsler geometry.

We shall denote

$$R_{\gamma\beta\alpha}^\kappa = K_{\gamma\beta\alpha}^\kappa$$

for all (β, α) except when $(\beta, \alpha) = (j, i)$, $(\beta, \alpha) = (b, i)$, $(\beta, \alpha) = (i, b)$, $(\beta, \alpha) = (q, i)$ and $(\beta, \alpha) = (i, q)$. In these cases we have

$$\begin{aligned}
 R_{\gamma ji}^\kappa &= K_{\gamma ji}^\kappa - \mathcal{N}_{ij}^c C_{\gamma c}^\kappa - \mathcal{M}_{ij}^q L_{\gamma q}^\kappa, \\
 R_{\gamma ib}^\kappa &= K_{\gamma ib}^\kappa + \partial_b \mathcal{N}_i^c C_{\gamma c}^\kappa, \\
 R_{\gamma bi}^\kappa &= K_{\gamma bi}^\kappa - \partial_b \mathcal{N}_i^c C_{\gamma c}^\kappa, \\
 R_{\gamma iq}^\kappa &= K_{\gamma iq}^\kappa + \partial_q \mathcal{M}_i^p L_{\gamma p}^\kappa, \\
 R_{\gamma qi}^\kappa &= K_{\gamma qi}^\kappa - \partial_q \mathcal{M}_i^p L_{\gamma p}^\kappa.
 \end{aligned}
 \tag{3.9}$$

From (3.7), (3.8) and (3.9) it is obvious that

$$R_{\gamma\beta\alpha}^\kappa = -R_{\gamma\alpha\beta}^\kappa.$$

We can write (3.6) in the form:

$$\begin{aligned}
 R(X, Y)Z = & \left[\frac{1}{2} K_{\gamma \beta \alpha}^{\kappa} (X^{\alpha} Y^{\beta} - Y^{\alpha} X^{\beta}) \right. \\
 & - \frac{1}{2} (\mathcal{N}_{ij}^c C_{\gamma c}^{\kappa} + \mathcal{M}_{ij}^q L_{\gamma q}^{\kappa}) (X^i Y^j - Y^i X^j) \\
 & - \frac{1}{2} \partial_b \mathcal{N}_i^c C_{\gamma c}^{\kappa} (X^i Y^b - Y^i X^b) \\
 & + \frac{1}{2} \partial_b \mathcal{N}_i^c C_{\gamma c}^{\kappa} (Y^i X^b - X^i Y^b) \\
 & - \frac{1}{2} \partial_q \mathcal{M}_i^p L_{\gamma p}^{\kappa} (X^i Y^q - Y^i X^q) \\
 & \left. + \frac{1}{2} \partial_q \mathcal{M}_i^p L_{\gamma p}^{\kappa} (Y^i X^q - X^i Y^q) \right] Z^{\gamma} \delta_{\kappa}
 \end{aligned}
 \tag{3.11}$$

For $(\beta, \alpha) = (j, i)$ the sum of the first and the second line in (3.11) is equal to $\frac{1}{2} R_{\gamma ji}^{\kappa} (X^i Y^j - Y^i X^j)$, for $(\beta, \alpha) = (b, i)$ the sum of the first and the third line in (3.11) is equal to $\frac{1}{2} R_{\gamma bi}^{\kappa} (X^i Y^b - Y^i X^b)$ etc.

From (3.8)–(3.11) follows

Theorem 3.1. *The curvature tensor of the second order gauge connection ∇ has the form*

$$R(X, Y)Z = \frac{1}{2} R_{\gamma \beta \alpha}^{\kappa} (X^{\alpha} Y^{\beta} - Y^{\alpha} X^{\beta}) Z^{\gamma} \delta_{\kappa},
 \tag{3.12}$$

where the components of R are determined by (3.8) and (3.9).

Formula (3.12) is short and elegant, but the explicit form of the curvature tensor is much longer, for instance if $(\beta, \alpha) = (b, i)$ then from (3.7) and (3.9) we have:

$$\begin{aligned}
 R_{\gamma bi}^{\kappa} = & \delta_i C_{\gamma b}^{\kappa} - F_{\gamma i}^{\theta} C_{\theta b}^{\kappa} - \partial_b F_{\gamma i}^{\kappa} + C_{\gamma b}^{\theta} F_{\theta i}^{\kappa} - \partial_b \mathcal{N}_i^c C_{\gamma c}^{\kappa} \\
 = & \delta_i C_{\gamma b}^{\kappa} - F_{\gamma i}^k C_{kb}^{\kappa} - F_{\gamma i}^c C_{cb}^{\kappa} - F_{\gamma i}^r C_{rb}^{\kappa} \\
 & - \partial_b F_{\gamma i}^{\kappa} + C_{\gamma b}^k F_{ki}^{\kappa} + C_{\gamma b}^c F_{ci}^{\kappa} + C_{\gamma b}^r F_{ri}^{\kappa} - \partial_b \mathcal{N}_i^c C_{\gamma c}^{\kappa}.
 \end{aligned}$$

4. Ricci identities for ∇

From (2.6) it follows

$$(4.1) \quad \begin{aligned} \nabla_X \nabla_Y Z &= [(Z^\gamma_{|\beta}) Y^\beta]_{|\alpha} X^\alpha \delta_\gamma \\ &= (Z^\gamma_{|\beta|\alpha} Y^\beta + Z^\gamma_{|\beta} Y^\beta_{|\alpha}) X^\alpha \delta_\gamma. \end{aligned}$$

From (2.6), (3.3) and (3.4) we obtain

$$(4.2) \quad \nabla_{[X,Y]} Z = Z^\gamma_{|\beta} [X, Y]^\beta \delta_\gamma = A + B,$$

where

$$(4.3) \quad \begin{aligned} A &= Z^\gamma_{|\beta} [X^\alpha (\partial_\alpha Y^\beta) - Y^\alpha (\partial_\alpha X^\beta)] \delta_\gamma \\ &= Z^\gamma_{|\beta} [X^\alpha Y^\beta_{|\alpha} - Y^\alpha X^\beta_{|\alpha} - (\Gamma^\beta_{\theta\alpha} - \Gamma^\beta_{\alpha\theta}) X^\alpha Y^\theta] \delta_\gamma \end{aligned}$$

$$(4.4) \quad \begin{aligned} B &= X^i Y^j [Z^\gamma|_c \mathcal{N}_i^c{}_j + Z^\gamma|_q \mathcal{M}_i^q{}_j] \delta_\gamma \\ &\quad + (X^i Y^b - Y^i X^b) Z^\gamma|_c (\partial_b \mathcal{N}_i^c) \delta_\gamma \\ &\quad + (X^i Y^q - Y^i X^q) Z^\gamma|_p (\partial_q \mathcal{M}_i^p) \delta_\gamma. \end{aligned}$$

Taking into account (2.14) and (2.15) we obtain

$$(4.5) \quad A + B = \left[Z^\gamma_{|\beta} \left(X^\alpha Y^\beta_{|\alpha} - Y^\alpha X^\beta_{|\alpha} \right) - Z^\gamma_{|\kappa} T^\kappa_{\beta\alpha} X^\alpha Y^\beta \right] \delta_\gamma.$$

From (4.1), (4.2) and (4.5) we obtain

$$(4.6) \quad \begin{aligned} R(X, Y) Z &= (Z^\gamma_{|\beta|\alpha} - Z_{|\alpha|\beta} + Z^\gamma_{|\kappa} T^\kappa_{\beta\alpha}) X^\alpha Y^\beta \delta_\gamma \\ &= \frac{1}{2} (Z^\gamma_{|\beta|\alpha} - Z^\gamma_{|\alpha|\beta} + Z^\gamma_{|\kappa} T^\kappa_{\beta\alpha}) (X^\alpha Y^\beta - Y^\alpha X^\beta) \delta_\gamma. \end{aligned}$$

From (4.6) and (3.12) it follows:

Theorem 4.1. *The Ricci equations for the second order gauge connection ∇ have the form:*

$$(4.7) \quad Z^\gamma_{|\beta|\alpha} - Z^\gamma_{|\alpha|\beta} + Z^\gamma_{|\kappa} T^\kappa_{\beta\alpha} = R^\gamma_{\kappa\beta\alpha} Z^\kappa.$$

(4.7) contains 3^3 types of Ricci equations, because each Greek index may be an element from one of the sets: $\{i, j, h, k, l\}$, $\{a, b, c, d, e\}$, $\{p, q, r, s, t\}$.

For $(\beta, \alpha) = (j, i)$ (4.7) becomes

$$\begin{aligned} Z^\gamma_{|j|i} - Z^\gamma_{|i|j} + Z^\gamma_{|k} T^k_{ji} + Z^\gamma|_c T^c_{ji} + Z^\gamma|_p T^p_{ji} \\ = R^\gamma_{kji} Z^k + R^\gamma_{cji} Z^c + R^\gamma_{pji} Z^p, \end{aligned}$$

for $(\beta, \alpha) = (p, i)$ (4.7) takes the form

$$\begin{aligned} Z^\gamma \|_{p|i} - Z^\gamma \|_i \|_p + Z^\gamma |_{|k} T_{p\ i}^k + Z^\gamma |_c T_{p\ i}^c + Z^\gamma \|_r T_{p\ i}^r \\ = R_{k\ pi}^\gamma Z^k + R_{c\ pi}^\gamma Z^c + R_{r\ pi}^\gamma Z^r, \text{ e.t.c.} \end{aligned}$$

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